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AN

EXAMINATION MANUAL

IN

PLANE GEOMETRY.

BY

G. A. WENTWORTH

AND

G. A. HILL.

BOSTON, U.S.A.
PUBLISHED BY GINN & COMPANY.
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PREFACE.

This book consists of a chapter containing definitions of the important terms used in Plane Geometry, and a numbered list of the standard propositions (arranged as in Wentworth's Plane Geometry); a short chapter on the methods of attacking and proving original theorems; a short chapter on the methods of attacking and solving original problems; a series of one-hour examination papers on each of the five books of Plane Geometry, each paper made up partly of text-book propositions, referred to by their numbers in the first chapter, and partly of originals; and some recent entrance examination papers in Plane Geometry.

The aim of the authors has been to give some elementary but much needed instruction in the art of handling original theorems and problems; and to supply a series of graded test-papers in Geometry which can be used not merely as *tests* of knowledge actually obtained, but also as a *means* of developing and strengthening the power to originate and carry on a logical train of thought; in other words, as a means of gaining the great object for which Geometry ought to be studied.

Notice of errors will be thankfully received by the authors.

EXAMINATION MANUAL.

PLANE GEOMETRY.

CHAPTER I.

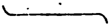
Definitions and Propositions of Plane Geometry.

SIGNS AND ABBREVIATIONS.

+	for increased by.	▱s	for parallelograms.
−	“ diminished by.	pt.	“ point.
×	“ multiplied by.	st.	“ straight.
÷	“ divided by.	rt.	“ right.
=	“ is (are) equal to.	adj.	“ adjacent.
>	“ is greater than.	isos.	“ isosceles.
<	“ is less than.	equil.	“ equilateral.
⊥	“ perpendicular.	ax.	“ axiom.
⊥s	“ perpendiculars.	hyp.	“ hypothesis.
∥	“ parallel.	const.	“ construction.
∠	“ angle.	def.	“ definition.
∠s	“ angles.	∴	“ therefore.
△	“ triangle.	ext.-int.	for exterior interior.
△s	“ triangles.	alt.-int.	“ alternate interior.
▱	“ parallelogram.	sup.-adj.,	supplementary adjacent.

DEFINITIONS.

1. A *Point* · has position but no magnitude.

2. A *Line*  has length but neither breadth nor thickness.

3. A *Straight Line* ———————— is a line described by a point moving always in the same direction.

4. A *Curved Line* is a line no part of which is straight.

5. *Equal Lines* are two lines which can be placed so that they coincide from end to end.

6. A *Common Measure* of two lines is a line which is contained an integral number of times in each of the two lines.

7. The *Ratio* of two lines is their relative magnitude when expressed by the quotient of the first divided by the second. That is, if a common measure is contained a times in the first line and b times in the second line, the ratio of the lines is the fraction $\frac{a}{b}$, or as also written, $a : b$. a is called the *antecedent*, b the *consequent* of the ratio.

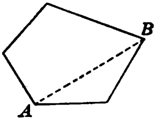
8. A *Unit of Length* is a standard length chosen as a common measure for all lines; as a foot, an inch, a yard.

9. The *Length* of a line is the number of units of length contained in the line; as 6 feet, $3\frac{1}{2}$ inches.

10. A *Surface* has length and breadth but no thickness.

11. A *Plane Surface* is a surface such that a straight line joining any two points in it lies wholly in the surface; as the surface of a slate.

12. A *Plane Figure* is a figure all points of which are in the same plane.



13. A *Polygon* is a plane figure bounded by straight lines. The straight lines are called its *sides*, their sum its *perimeter*, their points of meeting its *vertices*. If no side, however far produced, cuts another side, the polygon is called *convex*.

14. A *Diagonal* of a polygon is a straight line joining any two vertices of a polygon which are not adjacent.

15. A *Triangle* is a polygon having 3 sides.

16. A *Quadrilateral* is a polygon having 4 sides.

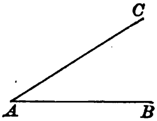
17. A *Pentagon* is a polygon having 5 sides.

18. A *Hexagon* is a polygon having 6 sides.

19. An *Octagon* is a polygon having 8 sides.

20. A *Decagon* is a polygon having 10 sides.

21. A *Dodecagon* is a polygon having 12 sides.



22. An *Angle* is formed by two straight lines which meet. The lines are called the *sides* of the angle; their point of meeting the *vertex*. The angle in the figure is named angle *BAC*.

23. *Equal Angles* are angles which can be so placed that they coincide, and form one angle.

24. The *Magnitude* of an angle is the amount of rotation about the vertex necessary to make one of the sides coincide with the other side.

25. An *Angle of one Revolution* is an angle described by a straight line which revolves about one end till it returns to its first position.

26. A *Degree* (°) is an angle equal to $\frac{1}{360}$th of a revolution.

27. A *Straight Angle* is an angle whose sides form a straight line; therefore equal to half an angle of revolution, or 180 degrees.

28. A *Right Angle* is an angle equal to half a straight angle, or 90 degrees.

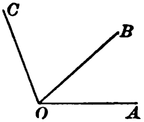
29. An *Acute Angle* is an angle less than a right angle.

30. An *Obtuse Angle* is an angle greater than a right angle, but less than a straight angle.

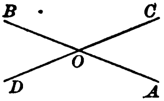
31. *Supplementary Angles* are two angles whose sum equal 180 degrees.

32. *Complementary Angles* are two angles whose sum equal 90 degrees.

33. *Adjacent Angles* are two angles having a common vertex and a common side between them.

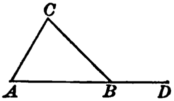


34. *Vertical Angles* are two angles having a common vertex, and the sides of one opposite in direction to those of the other.



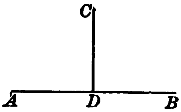
35. *Adjacent Angles* of a polygon are any two angles of the polygon having one side common.

36. An *Exterior Angle* of a polygon is an angle formed by producing a side of a polygon; as angle *CBD*.



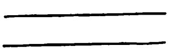
37. The *Bisector of an Angle* is a straight line dividing an angle into two equal parts.

38. *Perpendicular Lines* are straight lines which form a right angle. Thus, if the angle *CDA* is a right angle, the line *CD* is perpendicular to *AB*, and *AB* is perpendicular to *CD*.



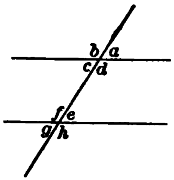
39. The *Right Bisector* of a straight line is the perpendicular erected at the middle point of the straight line.

40. *Parallel Lines* are straight lines which never meet, however far produced.



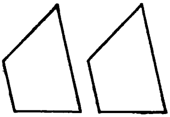
41. A *Transversal* is any straight line cutting two or more straight lines.

42. *Exterior-interior Angles* are the four pairs of angles, *a* and *e*, *b* and *f*, *h* and *d*, *g* and *c*, made by the transversal in the figure.



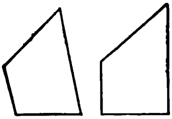
43. *Alternate-interior Angles* are the two pairs of angles, *c* and *e*, *d* and *f*, in the figure.

44. *Equal Polygons* are polygons that can be placed so that they coincide in all their parts (sides and angles).

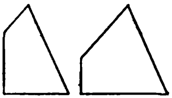


45. The *Homologous parts* of equal polygons are the parts which are similarly placed; in other words, the pairs of equal sides and the pairs of equal angles.

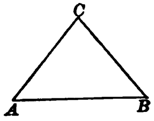
46. *Mutually Equilateral Polygons* are polygons whose sides, taken pair by pair in the same order, are equal.



47. *Mutually Equiangular Polygons* are polygons whose angles, taken pair by pair in the same order, are equal.



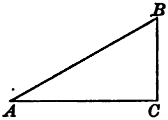
48. An *Isosceles Triangle* is a triangle having two equal sides. The equal sides are called the *legs*, the third side the *base*, the vertex opposite the base the *vertex* of the triangle.



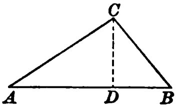
49. An *Equilateral Triangle* is a triangle having three equal sides.

50. A *Scalene Triangle* is a triangle having no two sides equal.

51. A *Right Triangle* is a triangle having a right angle. The side (*AB*) opposite the right angle is called the *hypotenuse;* the other two sides are called the *legs.*

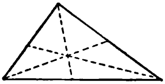


52. The *Base* of a triangle is the side of a triangle upon which it is supposed to stand. In an isosceles triangle the side unequal to the others is usually taken as the base.



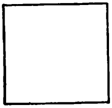
53. The *Altitude* of a triangle is the perpendicular from the vertex to the base, or base produced.

54. The *Medians* of a triangle are the lines joining the vertices of a triangle to the middle points of the opposite sides.

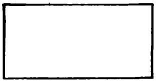


55. A *Parallelogram* is a quadrilateral whose opposite sides are parallel.

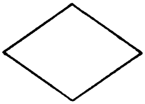
56. A *Square* is a parallelogram whose sides are equal and angles right angles.



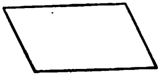
57. A *Rectangle* is a parallelogram whose adjacent sides are unequal, and whose angles are right angles.



58. A *Rhombus* is a parallelogram whose sides are equal, and whose angles are not right angles.



59. A *Rhomboid* is a parallelogram whose adjacent sides are unequal, and whose angles are not right angles.

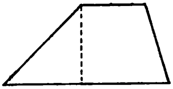


60. The *Bases* of a parallelogram are either pair of parallel sides.

61. The *Altitude* of a parallelogram is the distance between the bases.

62. The *Centre* of a parallelogram is the intersection of the diagonals.

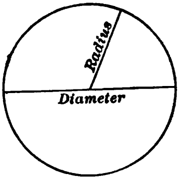
63. A *Trapezoid* is a quadrilateral having only two sides parallel. The parallel sides are called the *bases;* the other two sides the *legs;* the distance between the bases the *altitude.*



64. An *Isosceles Trapezoid* is a trapezoid having equal legs.

65. A *Trapezium* is a quadrilateral having no sides parallel.

66. A *Circumference* is a curve described by a point so moving that its distance from a fixed point remains always the same. The fixed point is called the *centre.*



67. A *Circle* is the portion of a plane surface bounded by a circumference.

68. A *Radius* is any straight line drawn from the centre to the circumference.

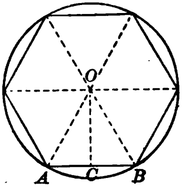
69. A *Diameter* is any straight line drawn through the centre and limited by the circumference.

90. The *Area* of a plane figure is the number of units of surface contained in the figure ; as 4 sq. ft., 9¼ sq. in.

91. *Equivalent Figures* are figures having the same area.

92. A *regular Polygon* is a polygon having all its sides equal, and all its angles also equal.

93. The *Centre* of a regular polygon is a point within a regular polygon equidistant from all its vertices. This point is also equidistant from all the sides.



94. The *Radius* of a regular polygon is the distance from the centre to any vertex (or the radius of the circumscribed circle).

95. The *Apothem* of a regular polygon is the distance from the centre to any one of the sides (or the radius of the inscribed circle).

96. *Similar Arcs, Sectors, Segments* are arcs, sectors, or segments in different circles corresponding to equal central angles.

97. A *Postulate* is an operation or construction assumed to be possible.

98. An *Axiom* is a proposition accepted as true without proof.

99. A *Theorem* is a proposition to be proved.

100. A *Corollary* is a proposition, the truth of which follows easily from truths already known.

101. A *Problem* is a proposition requiring the construction of a figure that shall satisfy given conditions.

PROPOSITIONS.

102. *Postulate.* Let it be granted that a figure can be moved from one place to another without change of size or shape.

103. *Postulate.* Let it be granted that a straight line can be drawn from any point to any other point.

104. *Postulate.* Let it be granted that a straight line can be produced as far as we please.

105. *Postulate.* Let it be granted that a circumference or arc can be described with any point as centre and any length as radius.

106. *Axiom.* Through a given point in a given direction only one straight line can be drawn.

107. *Axiom.* Through two given points only one straight line can be drawn.

108. *Axiom.* A straight line is the shortest line between two points.

109. *Axiom.* Through a given point only one straight line can be drawn parallel to a given straight line.

110. *Axiom.* For any quantity its equal can be substituted.

111. *Axiom.* If equals be added to equals the sums are equal.

112. *Axiom.* If equals be taken from equals the remainders are equal.

113. *Axiom.* The doubles of equals are equal; and the doubles of unequals are unequal.

114. *Axiom.* The halves of equals are equal; and the halves of unequals are unequal.

115. *Axiom.* If equals be added to unequals the sums are unequal.

116. *Axiom.* If equals are taken from unequals the remainders are unequal.

117. *Axiom.* If unequals be taken from equals the remainders are unequal; and the greater remainder comes from subtracting the smaller of the unequals.

118. *Axiom.* The whole is greater than a part.

BOOK I. — THE STRAIGHT LINE.

119. *Theorem.* All straight angles are equal.

120. *Corollary.* All right angles are equal.

121. *Corollary.* At a given point of a given straight line only one perpendicular can be drawn.

122. *Corollary.* Adjacent angles whose exterior sides form a straight line are supplementary.

123. *Corollary.* If two supplementary adjacent angles are equal, each is a right angle.

124. *Corollary.* The supplements of equal angles are equal.

125. *Corollary.* The complements of equal angles are equal.

126. *Theorem.* Vertical angles are equal.

127. *Theorem.* From a given point only one perpendicular can be drawn to a given straight line.

128. *Theorem.* Perpendiculars to the same straight line are parallel.

129. *Theorem.* A transversal perpendicular to one of two parallels is also perpendicular to the other.

130. *Theorem.* If two parallels are cut by a transversal, the alternate-interior angles are equal by pairs, and the exterior-interior angles are also equal by pairs.

131. *Theorem.* If two parallels are cut by a transversal, the interior angles on the same side of the transversal are supplementary.

132. *Theorem.* If two straight lines are so cut by a transversal that a pair of alternate-interior angles or a pair of exterior-interior angles are equal, the lines are parallel.

133. *Theorem.* Two straight lines each parallel to a third straight line are parallel to each other.

134. *Corollary.* The sum of two sides of a triangle is greater than the third side (108).

135. *Theorem.* The sum of two sides of a triangle is greater than the sum of the lines joining any point within the triangle to the ends of the third side.

136. *Theorem.* The sum of the angles of a triangle is equal to 180 degrees.

137. *Corollary.* An exterior angle of a triangle is equal to the sum of the two opposite interior angles.

138. *Corollary.* The sum of the acute angles of a right triangle is equal to 90 degrees.

139. *Corollary.* If two triangles have two pairs of equal angles, the remaining angles are also equal.

140. *Theorem.* A perpendicular is the shortest distance from a point to a straight line.

141. *Corollary.* The shortest line from a point to a straight line is perpendicular to the straight line.

142. *Theorem.* Two oblique lines drawn from a point in a straight line cutting off equal distances from the foot of the perpendicular are equal.

143. *Theorem.* Every point in the right bisector of a straight line is equidistant from the ends of the line.

144. *Theorem.* Every point which is not in the right bisector of a straight line is unequally distant from the ends of the line.

145. *Theorem.* Two triangles are equal if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.

146. *Theorem.* Two triangles are equal if two angles and a side of one are equal respectively to two angles and the homologous side of the other.

147. *Theorem.* In an isosceles triangle the angles opposite the equal sides are equal.

148. *Corollary.* An equilateral triangle is equiangular, and each angle is equal to 60 degrees.

149. *Theorem.* If two angles of a triangle are equal, the opposite sides are also equal.

150. *Corollary.* An equiangular triangle is equilateral.

151. *Theorem.* If two sides of a triangle are unequal, the opposite angles are also unequal, and the greater angle lies opposite the greater side.

152. *Theorem.* If two angles of a triangle are unequal, the opposite sides are also unequal, and the greater side lies opposite the greater angle.

153. *Theorem.* Two triangles are equal if the three sides of the one are equal respectively to the three sides of the other.

154. *Theorem.* Two right triangles are equal if a leg and the hypotenuse of the one are equal respectively to a leg and the hypotenuse of the other.

155. *Theorem.* Every point in the bisector of an angle is equidistant from the sides of the angle.

156. *Theorem.* Every point equidistant from the sides of an angle lies in the bisector of the angle.

157. *Theorem.* The opposite sides of a parallelogram are equal, and the opposite angles are also equal.

158. *Theorem.* A diagonal divides a parallelogram into two equal triangles.

159. *Corollary.* Parallels are everywhere equally distant.

160. *Theorem.* The diagonals of a parallelogram bisect each other.

161. *Theorem.* If two opposite sides of a quadrilateral are equal and parallel, the other two sides are equal and parallel, and the figure is a parallelogram.

162. *Theorem.* Rectangles having equal bases and equal altitudes are equal.

163. *Theorem.* If parallels divide one transversal into equal parts, they divide all transversals into equal parts.

164. *Theorem.* A straight line which bisects one side of a triangle and is parallel to another side bisects the third side.

165. *Theorem.* A straight line which bisects two sides of a triangle is parallel to the third side.

166. *Theorem.* The sum of the angles of a polygon of n sides is equal to $(n - 2)$ 180 degrees.

167. *Corollary.* The sum of the angles of a quadrilateral is equal to 360 degrees.

BOOK II. — THE CIRCLE.

168. *Theorem.* A diameter bisects a circle and its circumference.

169. *Theorem.* A diameter is greater than any other chord.

170. *Theorem.* In the same circle or equal circles, equal central angles intercept equal arcs.

171. *Theorem.* In the same circle or equal circles, equal arcs correspond to equal central angles.

172. *Theorem.* In the same circle or equal circles, equal chords subtend equal arcs.

173. *Theorem.* In the same circle or equal circles, equal arcs are subtended by equal chords.

174. *Theorem.* The radius perpendicular to a chord bisects the chord and the arc subtended by it.

175. *Theorem.* The radius which bisects a chord is perpendicular to the chord.

176. *Theorem.* The perpendicular erected at the middle point of a chord passes through the centre.

177. *Theorem.* In the same circle or equal circles, equal chords are equally distant from the centre.

178. *Theorem.* In the same circle or equal circles, chords equally distant from the centre are equal.

179. *Theorem.* In the same circle or equal circles, the greater of two chords is at a less distance from the centre.

180. *Theorem.* In the same circle or equal circles, of two chords that which is at a less distance from the centre is the greater.

181. *Theorem.* A tangent to a circle is perpendicular to the radius drawn to the point of contact.

182. *Theorem.* A perpendicular to a radius at its end is a tangent to the circle.

183. *Theorem.* A perpendicular to a tangent at the point of contact passes through the centre.

184. *Theorem.* The tangents drawn from an exterior point to a circle are equal.

185. *Theorem.* The line joining the centres of two intersecting circles is perpendicular to the common chord and bisects it.

186. *Theorem.* If two circles touch each other, the line joining their centres passes through their point of contact.

187. *Theorem.* If two variables are constantly equal and each approaches a limit, their limits are also equal.

188. *Theorem.* In the same circle or equal circles, two central angles have the same ratio as their intercepted arcs.

189. *Corollary.* A central angle is measured by the intercepted arc.

190. *Theorem.* An inscribed angle is measured by one-half the intercepted arc.

191. *Corollary.* An angle inscribed in a semicircle is a right angle.

192. *Corollary.* All angles inscribed in the same segment are equal.

193. *Theorem.* An angle formed by a tangent and a chord is measured by one-half the intercepted arc.

194. *Problem.* To find the ratio of two commensurable straight lines.

195. *Problem.* At a given point of a given straight line to erect a perpendicular to the line.

196. *Problem.* From a given point without a given straight line to draw a perpendicular to the line.

197. *Problem.* To bisect a given straight line.

198. *Problem.* To bisect a given arc.

199. *Problem.* To bisect a given angle.

200. *Problem.* To construct an angle equal to a given angle.

201. *Problem.* To draw through a given point a parallel to a given straight line.

202. *Problem.* To divide a given straight line into any number of equal parts.

203. *Problem.* Having given two angles of a triangle, to construct the third angle.

204. *Problem.* To construct a triangle, having given two sides and the included angle.

205. *Problem.* To construct a triangle, having given a side and two angles.

206. *Problem.* To construct a triangle, having given the three sides.

207. *Problem.* To construct a right triangle, having given the hypotenuse and one of the legs.

208. *Problem.* To describe a circumference through three given points.

209. *Problem.* To circumscribe a circle about a given triangle.

210. *Problem.* To inscribe a circle in a given triangle.

211. *Problem.* Through a given point to draw a tangent to a given circle (two cases).

212. *Problem.* Upon a given straight line to construct a segment capable of containing a given angle.

BOOK III. — SIMILAR FIGURES.

213. *Theorem.* If four quantities, a, b, c, d, are proportionals; that is, if $a : b = c : d$; then $ad = bc$.

214. *Theorem.* If four quantities, a, b, c, d, are such that $ad = bc$, then they are proportionals; that is, $a : b = c : d$.

215. *Theorem.* If b is a mean proportional between a and c; that is, if $a : b = b : c$; then $b^2 = ac$.

216. *Theorem.* If a, b, c are so related that $b^2 = ac$, then b is the mean proportional between a and c.

217. *Theorem.* If $a : b = c : d$, then

$$a : c = b : d$$
$$a \pm \mathrm{b} : b = c \pm d : d$$
$$a : a \pm b = c : c \pm d$$
$$a + b : a - b = c + d : c - d.$$

218. *Theorem.* In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.

219. *Theorem.* A line parallel to one side of a triangle divides the other sides proportionally.

220. *Theorem.* A line which divides two sides of a triangle proportionally is parallel to the third side.

221. *Theorem.* The bisector of an angle of a triangle divides the opposite side into two segments proportional to the other two sides.

222. *Theorem.* The bisector of an exterior angle of a triangle meets the opposite side produced at a point the distances of which from the ends of the opposite side are proportional to the other two sides.

223. *Theorem.* Two triangles are similar if they have two pairs of equal angles.

224. *Theorem.* Two triangles are similar if they have a pair of equal angles and the including sides proportional.

225. *Theorem.* Two triangles are similar if their sides taken in order, pair by pair, are proportional.

226. *Theorem.* The homologous bases of similar triangles are to each other as their altitudes.

227. *Theorem.* Two similar polygons are composed of the same number of triangles, similar pair by pair, and similarly placed.

228. *Theorem.* Two polygons are similar if they are composed of the same number of triangles, similar pair by pair, and similarly placed.

229. *Theorem.* The perimeters of any two similar polygons have the same ratio as any two homologous sides.

230. *Theorem.* The perpendicular from the vertex of the right angle to the hypotenuse of a right triangle is a mean proportional between the segments of the hypotenuse.

231. *Theorem.* A leg of a right triangle is a mean proportional between the hypotenuse and the segment adjacent to the leg.

232. *Theorem.* The sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse.

233. *Theorem.* If two chords intersect, the product of the two parts of one is equal to the product of the two parts of the other.

234. *Theorem.* If two secants are drawn from a point to a circle, the product of one secant and its exterior segment is equal to the product of the other secant and its exterior segment.

235. *Theorem.* If from a point without a circle a tangent and a secant are drawn to the circle, the tangent is a mean proportional between the entire secant and its exterior segment.

236. *Problem.* To divide a given straight line into two parts having a given ratio.

237. *Problem.* To find the fourth proportional to three given straight lines.

238. *Problem.* To find the third proportional to two given straight lines.

239. *Problem.* To find the mean proportional between two given straight lines.

240. *Problem.* To divide a given straight line into two parts such that the greater part is a mean proportional between the whole line and the smaller part.

241. *Problem.* To divide a given straight line harmonically in a given ratio.

242. *Problem.* Upon a given straight line to construct a polygon similar to a given polygon.

BOOK IV. — Equivalent Figures.

243. *Theorem.* Rectangles having equal altitudes are to each other as their bases, and rectangles having equal bases are to each other as their altitudes.

244. *Theorem.* Two rectangles are to each other as the products of their bases and altitudes.

245. *Theorem.* The area of a rectangle is equal to the product of its base and its altitude.

246. *Theorem.* Parallelograms having equal bases and equal altitudes are equivalent.

247. *Corollary.* The area of a parallelogram is equal to the product of its base and its altitude.

248. *Theorem.* The area of a triangle is equal to half the product of its base and its altitude.

249. *Corollary.* Triangles having equal bases and equal altitudes are equivalent.

250. *Theorem.* The area of a trapezoid is equal to the product of half the sum of its bases and its altitude.

251. *Theorem.* If two triangles have a pair of equal angles, their areas are to each other as the products of the sides which form the equal angles.

252. *Corollary.* If two triangles have a pair of equal angles, and the products of the sides which form the equal angles are also equal, the triangles are equivalent.

253. *Theorem.* Similar triangles are to each other as the squares of their homologous sides.

254. *Theorem.* Similar polygons are to each other as the squares of their homologous sides.

255. *Problem.* To construct a square equivalent to the sum of two given squares.

256. *Problem.* To construct a square equivalent to the difference of two given squares.

257. *Problem.* To construct a square equivalent to a given parallelogram.

258. *Problem.* To construct a triangle equivalent to a given polygon.

259. *Problem.* To construct a square equivalent to a given triangle.

260. *Problem.* To construct a square having a given ratio to a given square.

BOOK V. — REGULAR POLYGONS.

261. *Theorem.* A circle can be circumscribed about every regular polygon, and also inscribed in every regular polygon.

262. *Theorem.* Regular polygons of the same number of sides are similar.

263. *Theorem.* The area of a regular polygon is equal to half the product of its perimeter and its apothem.

264. *Theorem.* The perimeters of similar regular polygons are to each other as their radii or as their apothems.

265. *Theorem.* The circumferences of two circles have the same ratio as their radii or as their diameters.

266. *Corollary.* The ratio of the circumference of a circle to the diameter is constant for all circles.

(The value of this ratio is denoted by π.)

267. *Corollary.* If the radius of a circle is denoted by r, the length of the circumference is equal to $2\pi r$.

268. *Theorem.* The area of a circle is equal to πr^2.

269. *Corollary.* The areas of circles are to each other as the squares of their radii, or as the squares of their diameters.

270. *Corollary.* The area of a sector has the same ratio to the area of the circle that the angle of the sector has to 360 degrees.

271. *Problem.* To inscribe in a given circle a regular polygon having half as many sides as a given inscribed regular polygon.

272. *Problem.* To inscribe in a given circle a regular polygon having twice as many sides as a given inscribed regular polygon.

273. *Problem.* To circumscribe about a given circle a regular polygon having the same number of sides as a given inscribed regular polygon.

274. *Problem.* To inscribe a square in a given circle.

275. *Problem.* To inscribe a regular hexagon in a given circle.

276. *Corollary.* The side of a regular inscribed hexagon is equal to the radius of the circle.

277. *Problem.* To inscribe a regular decagon in a given circle.

CHAPTER II.

The Proving of Theorems.

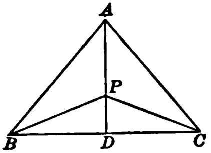
I. MAKING DEDUCTIONS.

The first thing a learner should do, if he wishes to acquire the power of proving original theorems in Geometry, is to learn by practice to make deductions correctly and rapidly from a given hypothesis or hypotheses by the aid of the standard propositions or theorems already proved.

We give below some examples for practice, the first two worked out.

1. HYPOTHESIS.

In $\triangle ABC$, $AB = AC$,
AD bisects $\angle A$,
P any pt. in AD.

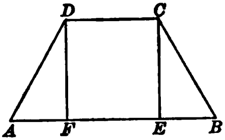


DEDUCTIONS.

$\triangle ABD = \triangle ACD$ . . . . (hyp., 145)
$BD = DC$, $\angle ABD = \angle ACD$, $\angle ADB = \angle ADC$, (45)
AD is $\perp$ to BC . . . . (123, 38)
$\triangle PBD = \triangle PCD$ . . . . (145)
$PB = BC$, $\angle PBD = \angle PCD$, $\angle BPD = \angle CPD$, (45)
$\triangle APB = \triangle APC$ . . . . (hyp., 153)
$\angle APB = \angle APC$, $\angle ABP = \angle ACP$ . . . . (45)

2. HYPOTHESIS.

In trapezoid *ABCD*,
AB is ∥ to *DC*, *AD* = *BC*,
CE is ⊥ to *AB*, *DF* is ⊥ to *AB*.



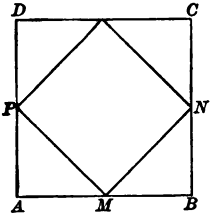
DEDUCTIONS.

CE and *DF* are ⊥ to *DC* . . . . (129)
CE is ∥ to *DF* . . . . (128)
CEFD is a rectangle . . . . (57)
CE = *DF*, *DC* = *FE* . . . . (157)
△ *ADF* = △ *BCE* . . . . (154)
∠ *A* = ∠ *B*, *AF* = *BE*, ∠ *ADF* = ∠ *BCE* . . . (45)
∠ *ADC* = ∠ *BCD* . . . . (111)

3. HYPOTHESIS.

A square *ABCD*.

M, *N*, *O*, *P*, the middle pts. of its sides.



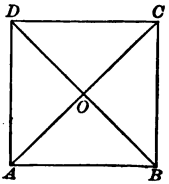
Make deductions. *Hints:* Compare the four ⊿ *APM*, etc.; try to find the values of the acute ∠s; examine the figure *MNOP*.

4. HYPOTHESIS.

A square *ABCD*.

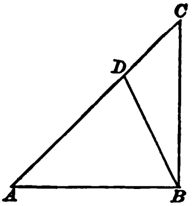
AC, *BD* the diagonals intersecting at *O*.

Make deductions.



5. Hypothesis.

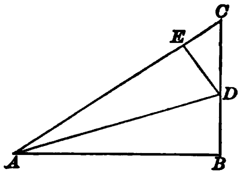
In $\triangle ABC$, $\angle ABC = 90°$,
$AB = BC$, $AD = AB$.



Make deductions. *Hint:* Try to find the value of all the $\angle$s in the figure.

6. Hypothesis.

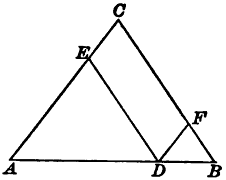
In $\triangle ABC$, $\angle ABC = 90°$,
AD bisects $\angle A$,
DE is $\perp$ to AC.



Make deductions. *Hints:* Compare $\triangle$s ABD, ADE; then BD and DE; then DE and DC; then BD and DC. Is $\angle CDE$ equal to any other $\angle$ in the figure?

7. Hypothesis.

In $\triangle ABC$, $AC = BC$,
D is any pt. in AB,
DE is $\parallel$ to BC, DF is $\parallel$ to AC.

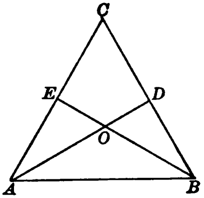


Make deductions. *Hints:* Examine the $\triangle$s ADE, DBF, and the figure $DFCE$. What is true of its perimeter?

8. Hypothesis.

In an equil. $\triangle ABC$,
AD is $\perp$ to BC, BE is $\perp$ to AC,
AD and BE intersect at O.

Make deductions.



9. $ABCD$ is an isosceles trapezoid. $AD = BC$, AB is $=$ to DC; CE is drawn $\parallel$ to AD, meeting AB at E. Make deductions.

10. ABC is an isos. $\triangle$. Upon the legs AB, AC, pts. D, E are taken so that $AD = AE$. BE and CD meet at O. Also DE is drawn. Make deductions.

11. ABC is an isos. rt. $\triangle$, $\angle A = 90°$, AD is $\perp$ to BC. Make deductions.

12. ABC is a rt. $\triangle$, $\angle A = 90°$, AD is $\perp$ to BC. Make deductions.

13. ABC is an equil. $\triangle$, M, N, P the middle pts. of its sides. The $\triangle\, MNP$ is drawn. Make deductions.

14. $ABCD$ is a rhombus; the diagonals AC, BD intersect at O. Make deductions.

15. $ABCD$ is a ▱; the diagonals AC, BD intersect at O; through O a st. line is drawn, meeting AB at E and CD at F. Make deductions.

16. $ABCD$ is an isosceles trapezoid; AB is $\parallel$ to CD, $AD = BC$; AD and BC produced meet at E; EF is $\perp$ to AB, meeting it at F, and cutting CD at G. Make deductions.

17. $ABCD$ is a quadrilateral; $AB = BC$, $CD = DA$; the diagonals AC, BD intersect at O. Make deductions.

18. AOB is the diameter of a circle; CD a chord $\parallel$ to AOB; AC, BD, AD, BC are drawn. Make deductions.

19. The pts. A, B, C, D, E, F divide a circumference into six equal arcs. The chords AB, BC, CD, DE, EF, FA are drawn; also the chords AC, CE, EA; also the radii OA, OC, OE. Make deductions.

II. METHODS OF PROOF.

There are *three* general methods of proving a theorem, the *synthetic* method, the *analytic* method, and the *indirect* method.

There are also special methods adapted to particular kinds of theorems; of these, the only ones important for the beginner are the method of *superposition* and the *algebraic* method.

1. The Synthetic Method.

Example 1. Prove that if two sides of a triangle are unequal, the opposite angles are unequal, and the greater angle lies opposite the greater side.

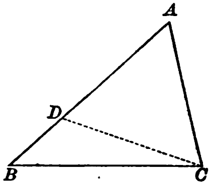
Hypothesis.

In $\triangle ABC$,

AB is $> AC$.

Conclusion.

$\angle ACB$ is $> \angle ABC$.



Proof. Upon AB take $AD = AC$, and join CD.

$\angle ACD = \angle ADC$ . . . . (147)

$\angle ACB$ is $> \angle ACD$ . . . . (118)

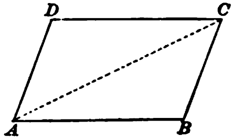
$\therefore \angle ACB$ is $> \angle ADC$ . . . . (110)

$\angle ADC = \angle ABC + \angle BCD$ . . . . (137)

$\therefore \angle ACB$ is $> \angle ABC + \angle BCD$ . . . . . . (110)

$\therefore \angle ACB$ is $> \angle ABC$ . . . . (118)

EXAMPLE 2. Prove that if the opposite sides of a quadrilateral are equal the figure is a parallelogram.



HYPOTHESIS.

In quadrilateral $ABCD$,
$AB = CD$, $BC = AD$.

CONCLUSION.

$ABCD$ is a ▱.

Proof. Draw AC.

In the $\triangle$ ABC, ACD,

AC is common,

$AB = CD$ . . . . (hyp.)

$BC = AD$ . . . . (hyp.)

$\therefore \triangle ABC = \triangle ACD$ . . . . (153)

$\therefore \angle BAC = \angle ACD$ . . . . (45)

$\therefore AB$ is $\parallel$ to CD . . . . (132)

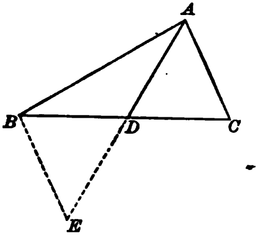
Similarly BC is $\parallel$ to AD.

$\therefore ABCD$ is a ▱ . . . . (55)

In the above examples we start from the hypothesis, and by combining it with truths already known, we proceed step by step until we arrive at the conclusion. A synthetic proof is a *putting together* of known truths in order to obtain a new truth.

Very often, as in the above examples, in order to combine known truths with the hypothesis, auxiliary lines must be drawn (CD in Example 1, AC in Example 2).

EXAMPLE 3. Prove that a median in a triangle is less than half the sum of the two adjacent sides.



HYPOTHESIS.

In $\triangle ABC$,
$BD = DC$.

CONCLUSION.

AD is $< \frac{1}{2}(AB + AC)$.

Proof. Produce AD, making $DE = AD$, and join BE.

In the $\triangle$ BED, ACD

$BD = DC$ . . . . (hyp.)
$DE = AD$ . . . . (const.)
$\angle BDE = \angle ADC$ . . . . (126)
$\therefore \triangle BED = \triangle ACD$ . . . . (145)
$\therefore BE = AC$ . . . . (45)
Now $AD + DE$ is $< AB + BE$ . . . . (134)
$\therefore 2AD$ is $< AB + AC$ . . . . (110)
$\therefore AD$ is $< \frac{1}{2}(AB + AC)$ . . . . (114)

In Example 3 the steps of the reasoning are easy to follow after the auxiliary lines DE, BE are drawn as described. But to one to whom the theorem is new, what is there to suggest the use of these particular lines? The synthetic method is logically a perfect method, but it gives no aid towards answering this question. And in many cases this question must be answered before the synthetic method or, indeed, any method of proof can be successfully used.

But let us suppose that suitable auxiliary lines have been drawn. It usually happens that, with the aid of truths already known, many deductions can be made that are superfluous if we wish to go by the shortest road from the hypothesis to the conclusion. This must be obvious to every one who has carefully worked out the exercises on pp. 24–27. How are we to avoid these unnecessary deductions in proving a new theorem, and make only those which carry us most directly from the hypothesis to the conclusion? This question the synthetic method wholly fails to answer. If you already know how to prove a theorem, the synthetic method is a very good method of convincing others that the theorem is true. But if you do not know how to prove a theorem, and want to discover the proof, the analytic method, which we now proceed to explain, is the method to be used.

2. The Analytic Method.

The reasoning in this method is of the following kind:

Suppose we wish to prove that the cube root of 512 is equal to 8. From the meaning of the term we know that the cube root of $512 = 8$ *if* $8 \times 8 \times 8 = 512$.

But $8 \times 8 \times 8 = 512$ (by actual multiplication).

Therefore, the cube root of $512 = 8$.

We can always prove a proposition A to be true by finding another proposition B of such a nature that *if* true A must be true also, and by then showing that B *is* true.

If the truth of B is not known, we try to continue the same line of reasoning with other propositions until we arrive at one which we know to be true. Then it follows that all the preceding propositions, including A which we wished to prove, are true.

EXAMPLE 1. The same as Example 1, page 28.

Analysis. We wish to prove that $\angle ACB$ is $> \angle ABC$.

In order to bring these $\angle$s into relation to each other, we take on AB a length $AD = AC$, and join DC. Then we know that $\angle ACD = \angle ADC$ (147), and that $\angle ADC = \angle ABC + \angle BCD$ (137).

Now $\angle ACB$ is $> \angle ABC$ if it is $> \angle ADC$ (137, 118)
And $\angle ACB$ is $> \angle ADC$ if it is $> \angle ACD$ . . (110)
But $\angle ACB$ is $> \angle ACD$ . . . . (118)
$\therefore \angle ACB$ is $> \angle ABC$.

The analysis is complete, and the proposition is proved.

EXAMPLE 2. The same as Example 2, page 29.

Analysis. We wish to prove that $ABCD$ is a ▱.

Now $ABCD$ is a ▱ if AB is $\parallel$ to CD, and AD is $\parallel$ to BC (55).

We now wish to prove that AB is $\parallel$ to CD.

If we join AC, we can proceed with the analysis as follows:

AB is $\parallel$ to BC if $\angle BAC = \angle ACD$ . . . . . (132)
$\angle BAC = \angle ACD$ if $\triangle ABC = \triangle ACD$ . . . . (45)
But $\triangle ABC = \triangle ACD$ . . . . (153)
For AC is common.
$AB = CD$ . . . . (hyp.)
$AD = BC$ . . . . (hyp.)

And the same reasoning shows that AD is $\parallel$ to BC.

$\therefore ABCD$ is a ▱.

Example 3. The same as Example 3, page 30.

Analysis. We wish to prove that AD is $< \frac{1}{2}(AB + AC)$.

AD is $< \frac{1}{2}(AB + AC)$ if $2AD$ is $< AB + AC$ (114)

This suggests producing AD by its own length to E, and joining BE; then $AE = 2AD$ and

$2AD$ is $< AB + AC$ if AE is $< AB + AC$ . . (110)

Now we know that AE is $< AB + BE$ . . . (134)

$\therefore AE$ is $< AB + AC$ if $AC = BE$ . . . . . (110)

And $AC = BE$ if $\triangle ACD = \triangle BDE$ . . . . (45)

But $\triangle ACD = \triangle BDE$ . . . . (145)

For $CD = BD$ . . . . (hyp.)

$AD = AE$ . . . . (const.)

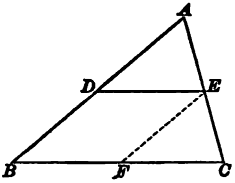
$\angle ADC = \angle BDE$ . . . . (126)

$\therefore AD$ is $< \frac{1}{2}(AB + AC)$.

A comparison of the analytic proofs just given of the above three examples with the synthetic proofs of the same examples on pages 28–30 will show that the analytic order of reasoning is exactly the reverse of the synthetic order. In synthesis we begin with the hypothesis and known propositions, and proceed step by step, deducing necessary consequences, till we reach the conclusion; in analysis we begin with the conclusion and reason backward, so to speak, till we arrive at the hypothesis or a proposition known to be true.

For the sake of illustrating as clearly as possible the nature and difference of the two methods, we will add two more examples of analysis, choosing for this purpose Propositions 164 and 165.

Example 4. Prove that a straight line which bisects one side of a triangle, and is parallel to another side, bisects the third side.



HYPOTHESIS.

In $\triangle ABC$, $AD = BD$,
DE is $\parallel$ to BC.

CONCLUSION.

$AE = EC$.

Analysis. $AE = EC$ if we can prove that they are homologous sides of equal $\triangle$.

AE is a side of the $\triangle ADE$; EC can be made the side of a $\triangle$ in three ways: by joining EB, by joining DC, or by drawing $EF \parallel$ to AB.

A very little reflection is sufficient to show that the third way alone will serve our purpose; so we draw $EF \parallel$ to AB.

Then $AE = EC$ if $\triangle ADE = \triangle EFC$ . . . (45)

Now $\angle A = \angle FEC$ . . . . (130)

$$\therefore \triangle ADE = \triangle EFC \text{ if} \begin{cases} (1)\ AD = EF, \text{ and} \\ (2)\ \angle ADE = \angle EFC. \end{cases} \quad (146)$$

(1) $AD = EF$ if each is equal to BD . . . (110)

But $AD = BD$, $EF = BD$ . . . . . . (hyp., 157)

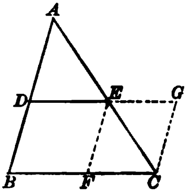
(2) $\angle ADE = \angle EFC$ if each is equal to $\angle B$. (110)

But $\angle ADE = \angle B$, $\angle EFC = \angle B$ . . . . . (130)

$$\therefore AE = EC.$$

The learner should now write out the synthetic proof, and observe how the general course of the reasoning is reversed.

EXAMPLE 5. Prove that a straight line which bisects two sides of a triangle is parallel to the third side.



HYPOTHESIS.

In $\triangle ABC$, $AD = BD$,
$AE = EC$.

CONCLUSION.

DE is $\parallel$ to BC.

Analysis. DE is $\parallel$ to BC if $\angle AED = \angle ECB$ (129).

Draw EF $\parallel$ to AB as in the last example.

Then $\angle AED = \angle ECB$ if $\triangle ADE = \triangle ECF$. (45)

Now, although these $\triangle$s are really equal, the hypothesis with the aid of the standard Propositions is not sufficient to prove them equal; so that the attempt to analyze the theorem with the line EF is a failure.

Draw CG $\parallel$ to AB and produce DE to meet it at G.

DE is $\parallel$ to BC if $BCGD$ is a ▱ . . . . (55)
$BCGD$ is a ▱ if $CG = BD$ . . . . (161)
And $CG = BD$ if each is equal to AD . . . (110)
Now $BD = AD$ . . . . (hyp.)
And $CG = AD$ if $\triangle CGE = \triangle ADE$ . . . (45)
But $\triangle CGE = \triangle ADE$ . . . . (146)
For $EC = AE$ . . . . (hyp.)
$\angle GEC = \angle AED$ . . . . (126)
$\angle ECG = \angle DAE$ . . . . (130)

$\therefore DE$ is $\parallel$ to BC.

The learner should write out the synthetic proof.

The analytic method might also be called the method of *reduction* or of *substitution.* It consists in finding a chain of propositions, beginning with the end of the chain — the proposition that we wish to prove — and going backwards, until we arrive at a proposition already proved or known to be true, the different propositions being so related that each one is a necessary consequence of the one which follows it in the chain; whence we infer that the first is a necessary consequence of the last, and true because the last is known to be true.

Which is the better method, analysis or synthesis?

The answer depends on the extent of our knowledge, and the purpose we have in view.

If we wish to discover how to prove a theorem, we should use the analytic method.

Analytic reasoning is much more likely to suggest useful auxiliary lines, and such lines are very often required before any relation between the theorem to be proved and a known theorem can be perceived. When such a relation is once clearly seen, analytic reasoning enables us to avoid useless steps, and to go in a direct line from the unknown theorem to others more and more nearly related to known theorems; and as soon as we reach a known theorem our work is done.

On the other hand, there is nothing in the synthetic method to suggest the use of suitable auxiliary lines; moreover at the very start we have to face the question: which one of the various possible deductions from the hypothesis will put us on the direct road to the conclusion? The synthetic method has no answer to give. And the question repeats itself at every step we take.

If we go on making deductions as they happen to occur to us, we may waste much time and labor without getting any nearer to the conclusion.

But if we know how to prove a theorem, and wish to present the proof in a concise, logical form, we should use the synthetic method.

Both methods are equally good as regards logical rigor, but the synthetic method has the advantage in conciseness, and is the method usually employed for proving theorems in text-books on Geometry.

Perhaps the best advice to the learner in working out original theorems is the following:

Discover how to prove the theorem by an analysis which need not be written out but simply thought out in the mind; then write out for the teacher the corresponding synthetic proof.

In making an analysis it is sometimes difficult to decide which to choose of two or more propositions that suggest themselves as possible new steps. In such cases the best way is to try one, and if it fails to answer the purpose, then to try another, and so proceed till the right one is found. But skill in making the selection is soon acquired by practice.

Success in this kind of work very often depends on drawing a suitable auxiliary line. The auxiliary lines most often useful are obtained by

(1) joining two given points;
(2) drawing a line parallel to a given line;
(3) drawing a line perpendicular to a given line;
(4) producing a line by its own length.

3. THE INDIRECT METHOD.

Suppose we wish to prove a proposition, which for the sake of brevity may be thus stated :

If A is B, then C is D.

We know that C must be either D or not D.

Assume that C is not D.

Now, if we can show that this assumption leads to a conclusion which we know to be false, either because it contradicts the hypothesis (if A is B) or some truth already proved, then we can infer that C is D.

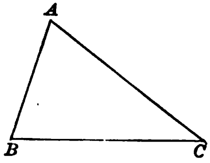
EXAMPLE 1. Prove that if one angle of a triangle is greater than another angle, the side opposite the first angle is greater than the side opposite the second angle.

HYPOTHESIS.

In $\triangle ABC$, $\angle B$ is $> \angle C$.

CONCLUSION.

AC is $> AB$.



Proof. AC is either *equal to AB, less than AB,* or *greater than AB.*

Of these suppositions, one and one only can be true.

Suppose that $AC = AB$; then $\angle B = \angle C$ . . (147)

But this contradicts the hypothesis.

$\therefore AC$ cannot be equal to AB.

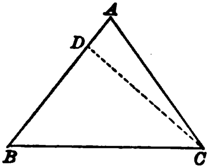
Suppose that AC is $< AB$; then $\angle B$ is $< \angle C$ (151)

This also contradicts the hypothesis.

$\therefore AC$ cannot be less than AB.

$\therefore AC$ is $> AB$.

Example 2. Prove that if two angles of a triangle are equal, the opposite sides are also equal.



HYPOTHESIS.

In $\triangle ABC$, $\angle B = \angle C$.

CONCLUSION.

$AB = AC$.

Proof. AB must be either ***greater than** AC*, ***less than** AC*, or *equal to AC*.

Suppose that AB is $> AC$. Upon BA take $BD = AC$ and join CD.

In the $\triangle$ BDC, BAC

BC is common.

$BD = AC$ . . . . (const.)

$\angle B = \angle C$ . . . . (hyp.)

$\therefore \triangle BDC = \triangle BAC$ . . . . (145)

But this would be asserting that a part is equal to the whole, and is therefore absurd . . . . . . (118)

$\therefore AB$ cannot be greater than AC.

In the same way we can show that AB cannot be less than AC.

$$\therefore AB = AC.$$

This peculiar mode of proving a theorem is known as the Indirect Method. It was a favorite method with the ancient geometers, and in certain cases it is a better method than any other (for example, Proposition 183); but in general, direct methods of proof are to be preferred.

It may throw more light on the nature of the indirect method if we enumerate the different theorems which can be *derived* from a given theorem by making certain changes in the hypothesis and conclusion.

The *contradictory* of a theorem is obtained by making the conclusion negative.

The *opposite* of a theorem is obtained by making both the hypothesis and the conclusion negative.

The *converse* of a theorem is obtained by transposing the hypothesis and the conclusion.

The following, using general symbols, are typical examples :

A theorem . . .	If A is B, then C is D.
Its contradictory .	If A is B, then C is not D.
Its opposite . .	If A is not B, then C is not D.
Its converse . .	If C is D, then A is B.

It is obvious that a theorem and its contradictory are so related that one of them must be true, and the other must be false.

The indirect method consists in showing that a theorem is true by proving its contradictory to be false.

4. Method of Superposition.

For examples of the use of this method, the learner is referred to the proofs of Propositions 145 and 146 given in the text-books.

The method is confined in its application to a few elementary propositions such as the two just mentioned. But for these few propositions it is the only method that can be used.

5. THE ALGEBRAIC METHOD.

This is a method specially adapted to the proof of numerical relations among the angles or the sides of a plane figure. The general rules for applying it are as follows :

Observe what numerical relations follow immediately from the hypothesis and from propositions already proved. State these relations in the form of equations. Then reduce the equations with the aid of the general axioms (110–118) until the desired result is obtained.

EXAMPLE 1. Prove that if one leg of an isosceles triangle be produced from the vertex by its own length, and the end joined to the end of the base, the joining line will be perpendicular to the base.

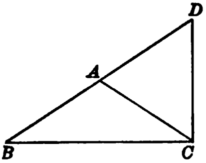
HYPOTHESIS.

In the isos. $\triangle ABC$.

$AB = AC = AD$.

CONCLUSION.

DC is $\perp$ to BC.



Proof. $\angle B + \angle D + \angle BCD = 180°$ . . . . . (136)

$\angle B = \angle ACB$, $\angle D = \angle ACD$ . . . (147)

By addition, $\angle B + \angle D = \angle BCD$ . . . (111)

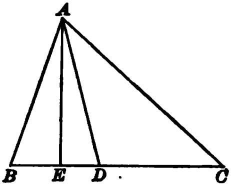
Substituting $\angle BCD$ for $\angle B + \angle D$ in the first equation

$2 \angle BCD = 180°$ . . . . (110)

$\therefore \angle BCD = 90°$ . . . . (114)

That is, DC is $\perp$ to BC.

Example 2. Prove that the angle between the altitude AE and the bisector AD of the angle A of a triangle ABC is equal to half the difference of the other two angles.



Hypothesis.

In $\triangle ABC$, AE is $\perp$ to BC,
$\angle BAD = \angle CAD$.

Conclusion.

$\angle EAD = \frac{1}{2}(\angle B - \angle C)$.

Proof. $\angle EAD + \angle ADE = 90°$ . . . . (138)
$\angle ADE = \angle C + \angle CAD$ . . . . (137)
$\angle ADE = \angle C + \frac{1}{2}\angle BAC$ . . . . . . (110)
$\angle BAC + \angle B + \angle C = 180°$ . . . . . . (136)
$\angle BAC = 180° - \angle B - \angle C$ . . . . . . (112)
$\angle ADE = \angle C + \frac{1}{2}(180° - \angle B - \angle C)$. (110)
$\angle EAD + \angle C + \frac{1}{2}(180° - \angle B - \angle C) = 90°$ (110)

Reducing, we have $\angle EAD = \frac{1}{2}(\angle B - \angle C)$.

III. THEOREMS CLASSIFIED.

All that will be attempted under this heading is to group theorems according to the nature of what is to be proved, and to give under each group a few suggestions intended to assist the learner in his efforts to discover a proof.

The classification is by no means complete; only the most common kinds are included. So great is the number and the variety of geometric theorems, that an exhaustive classification upon any rational principle is practically impossible.

Group 1. Theorems to prove two polygons equal.

If the polygons are triangles, apply one of the Propositions 145, 146, 153, 154. In other cases, use the method of superposition; or divide the polygons into triangles by drawing diagonals from one vertex, and then prove the triangles equal, pair by pair.

An analysis is seldom necessary.

Group 2. Theorems to prove two lines equal.

The analysis is usually effected by showing that the two lines are either (1) homologous sides of equal triangles, or (2) legs of an isosceles triangle, or (3) opposite sides of a parallelogram. If the two lines are chords in a circle, try to show that they are equally distant from the centre, or that they subtend equal arcs. Auxiliary lines are often necessary; the most useful have been mentioned already (see page 37).

Useful Propositions: 145, 146, 149, 153, 154, 157, 164, 173, 178.

Group 3. Theorems to prove two angles equal.

The algebraic method is sometimes sufficient (see page 41); if it cannot be applied, use the analytic method, and try to show that the two angles are either (1) homologous angles of equal triangles, or (2) base angles in an isosceles triangle. If the two angles are inscribed angles, try to show that they intercept equal arcs, or are inscribed in the same or equal segments.

Auxiliary lines may be required.

Useful Propositions: 123, 124, 125, 126, 130, 136, 137, 138, 139, 145, 146, 147, 148, 153, 154, 166, 167, 189, 190, 191, 192, 193.

Group 4. Theorems to prove two lines parallel or perpendicular to each other.

To prove two lines parallel, try by an analysis to reduce the theorem to Group 3 with the aid of Proposition 132 ; then proceed as in Group 3.

To prove two lines perpendicular to each other, try by analysis to reduce the theorem to Group 3 with the aid of Proposition 123 ; then proceed as in Group 3. Auxiliary lines are usually required.

Propositions 129, 133, 141, 175, 181, are sometimes useful.

Group 5. Theorems to prove two lines or angles unequal.

The Propositions most useful for this class of theorems are 108, 134, 135, 140, 151, 152.

Group 6. Theorems to prove that three lines are concurrent (i.e. pass through one point).

Assume a point of meeting for two of the lines, join this point to some point known to be in the third line, and try to prove that this joining line must coincide with the third line.

Group 7. Theorems to prove that three points are collinear (i.e. are in a straight line).

Join the middle point to the other two points, and try to prove that the two joining lines form a straight angle. Auxiliary lines are usually required.

Group 8. Theorems to prove two triangles similar.

Try to prove that they have either (1) two pairs of equal angles, or (2) one pair of equal angles and the including sides proportional (Propositions 223, 224).

Group 9. Theorems to prove that four lines are proportionals; or that the product of two lines is equal to the product of two other lines; or that a line is a mean proportional between two other lines.

Try to reduce the theorem to Group 8 by drawing suitable auxiliary lines ; then proceed as in Group 8.

Useful Propositions : 223, 224, 230, 231, 233, 234, 235.

Group 10. Theorems to prove two triangles equivalent.

Try to prove that they have equal bases and equal altitudes, and apply Proposition 249.

Sometimes Proposition 252 may be used.

CHAPTER III.

The Solving of Problems.

I. GEOMETRIC LOCI.

In solving problems, loci are often very useful. The learner should have a perfectly clear idea of what a locus is, and be familiar with those which are most important. For this reason a brief review of the subject will be given.

If a point so moves in a plane that it always satisfies a given condition, the line which it describes is called its *locus.*

In order to prove that a certain line is the locus of a point moving under a given condition, it is necessary to prove that *every point in the line satisfies the given condition.*

In order to prove that this line forms the entire locus of the point, we must prove *either* that *every point satisfying the given condition lies in the line* (the converse of the above proposition), *or* that *every point not in the line fails to satisfy the given condition* (the opposite of the above proposition).

It usually happens that if the first of these three propositions is true, the other two are also true.

The first loci for the learner to make himself familiar with are the following five :

1. The locus of a point which is at a given distance from a given point is the circumference of a circle, having the given point for centre, and the given distance for radius.

2. The locus of a point which is at a given distance from a given straight line consists of the two parallels to the line drawn, one on each side of the line, at the given distance from it.

3. The locus of a point equidistant from two given points is the right bisector of the straight line which joins the given points (143, 144).

4. The locus of a point equidistant from the sides of a given angle is the bisector of the angle (155, 156).

5. The locus of a point equidistant from two parallels is a straight line parallel to the given parallels and equidistant from them.

The following loci should be determined by the learner; and a proof that the locus has been correctly determined should be given in each case:

6. Locus of a point equidistant from two given intersecting straight lines. (The locus consists of two perpendicular lines.)

7. Locus of the centres of circles which pass through two given points.

8. Locus of the centres of circles which have a given radius and pass through a given point.

9. Locus of the centres of circles which have a given radius and touch a given straight line.

10. Locus of the centres of circles which touch a given straight line at a given point.

11. Locus of the centres of circles which touch two given parallel lines.

12. Locus of the centres of circles which touch two given intersecting straight lines.

13. Locus of the centres of circles which have a given radius and touch a given circle.

14. Locus of the centres of circles which touch a given circle at a given point.

15. Locus of the centres of circles which touch two given concentric circles.

16. Locus of the vertices of all right triangles having the same hypotenuse (191).

17. Locus of the vertices of all triangles having the same base and a given angle at the vertex (192).

18. Locus of the vertices of all triangles having the same base and a given altitude.

19. Locus of the middle points of parallel chords in a given circle.

20. Locus of the middle points of all chords in a given circle which have a given length.

21. Locus of the middle points of all chords drawn from a given point in the circumference of a given circle (175, 191).

22. Locus of the middle points of all chords in a given circle which pass through a common point.

23. Locus of the end points of tangents to a given circle that have a given length.

24. Locus of the middle points of straight lines drawn between two parallels.

25. Locus of the end points of all parallels of given length drawn from points of a given straight line on the same side of it.

26. Locus of all points from which tangents drawn to a given circle form the same angle.

27. Locus of the centres of all circles which cut equal chords from the sides of a given angle.

28. Locus of the centres of all circles which cut equal chords from two parallel straight lines.

29. AB is a straight line, and AC is drawn from A

perpendicular to any straight line passing through *B*. Find the locus of the middle point of *AC*.

30. A straight rod of given length slides between two straight rulers fastened together perpendicular to each other. What is the locus of its middle point?

31. A straight line *AB* moves so that *A* always touches a given circumference and *AB* is always parallel to a given straight line. What is the locus of *B*?

32. *AB* is a diameter of a circle, and *AC* any chord. *BD* is drawn perpendicular to the tangent at *C*, and meeting *AC* produced at *P*. Find the locus of *P* as *AC* turns about *A*. (Draw the radius *OC*, see 164, 175, 191).

33. *AB* is a fixed diameter of a circle, and the chord *AC* is produced to *P* so that *PC* is equal to *BC*. Find the locus of *P* as *AC* turns about *A*.

II. SYSTEMATIC SOLUTION OF A PROBLEM.

A geometric problem requires us, from certain given points, lines, or figures, to find others that shall satisfy certain given conditions.

The systematic solution of a problem is divisible into four parts: the *analysis*, the *construction*, the *proof*, and the *discussion*.

1. The Analysis.

If a problem is so simple that the solution is obvious by a direct application of a known theorem, there is nothing to do but to make the construction according to the theorem, and then give a synthetic proof that the construction is correct. Most of the fundamental problems given among the Propositions in Chapter I. are of this kind.

But problems are usually of a more difficult type. The application of known theorems to their solution is not immediate, and often far from obvious. To discover the mode of application is the first and most difficult part of the solution. The best way to attack such problems is by a method resembling the analytic proof of a theorem, and which may be termed the *analysis* of the problem.

Suppose the construction effected, and draw a figure representing all parts concerned, both given and required. Then study the relations among the parts with the aid of known theorems, and try to find some relation which will suggest the construction. If this attempt fails, introduce new relations by drawing auxiliary lines, and continue the same study of relations till a clue to the right construction is found.

Problems differ so widely in their nature, that more precise directions for making an analysis cannot be given, except by considering special cases. Some examples of analysis will be given later. It often happens that an analysis of a problem may be successfully made in several different ways.

2. THE CONSTRUCTION.

Besides a pen or pencil, no instruments should be used in making the construction, except a *ruler* for drawing straight lines and a *pair of compasses* for describing the circumferences and arcs of circles.

Given lines should be drawn *thin;* required lines should be drawn *thicker;* auxiliary lines should be ·awn *dotted.*

3. The Proof.

The proof should be a synthetic proof (see page 28). The use of the proof is simply to verify the correctness of the analysis, and consequently of the construction. Generally speaking such verification is not needed. It is better, as a rule, to omit the proof, and devote the time thus saved to the solution of new problems.

4. The Discussion.

The object of the discussion is to examine the problem with respect to its being determinate, indeterminate, or impossible. A problem is *determinate* if it has one, two, or any finite number of solutions ; *indeterminate* if it has an infinite number of solutions ; *impossible* if it has no solution.

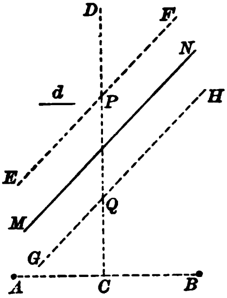
It often happens that a problem is determinate for certain relative positions or magnitudes of the given parts, and indeterminate or impossible for other positions or other magnitudes.

The discussion should take into account these variations of position and magnitude, and trace their effect on the character of the problem.

The best way to make rapid progress in the power to solve problems, is to begin by working out a few systematically and completely, and then to omit both construction and proof, confining the attention entirely to the analysis and discussion.

III. ANALYSIS OF A PROBLEM. EXAMPLES.

1. Find a point equidistant from two given points, and at a given distance from a given straight line.



Analysis. Suppose the problem solved, *A* and *B* the given points, *MN* the given line, *d* the given distance, *P* and *Q* points which satisfy the given conditions.

To satisfy the first condition, the required point must be in the right bisector *CD* of the line joining the given points *A* and *B* (Locus 3).

To satisfy the second condition the required point must be in one of the parallels *EF*, *GH* to the line *MN*, drawn at the given distance from it (Locus 2).

Therefore, the required point being situated in *both* loci, must be their intersection.

This method of analysis is known as the Method of Loci, and is applicable to most problems which require the determination of a point.

Construction. This is effected by the direct use of Propositions 195 and 201. We shall not occupy space by giving it in detail.

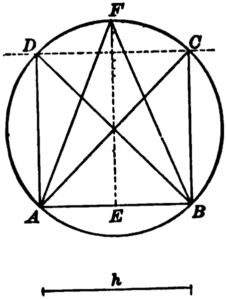
Proof. A proof is superfluous.

Discussion. If *MN* cuts *CD* as shown in the figure there are *two* solutions, namely, the points *P* and *Q*.

If *MN* is || to *CD*, the problem is in general impossible; but if *MN* is || to *CD* and at the given distance *d* from *CD*, the problem is indeterminate.

2. Construct a triangle, having given the base, the angle opposite the base, and the altitude.

Analysis. Suppose the problem solved, ABC the required △, AB the given base, and h the given altitude.



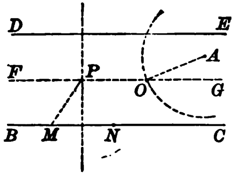
Two vertices A, B of the required △ are given. One locus for C is the line DC drawn ∥ to AB at the distance h from AB. Another locus is the arc of a segment capable of containing the given ∠ (Locus 17). The intersection of these loci determines C.

Construction. Apply Propositions 201 and 212.

Discussion. If h is $<$ the height EF of the segment $ADCB$, there are *two* solutions, the △s ABC, ABD; if $h = EF$ there is *one* solution, the isosceles △ ABF; if h is $>$ EF, the problem is impossible.

3. Construct a circle which shall pass through a given point, and cut chords of equal given length from two given parallels.

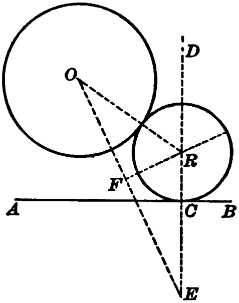
Analysis. Suppose the problem solved, A the given point, BC and DE the given parallels, MN the given length, O the centre of the required circle. One locus for O is the line FG ∥ to BC and equidistant from BC and DE (Locus 28). Draw the right bisector of MN; this determines



the radius *MP* of the circle. With *A* as centre and *MP* for radius, describe an arc cutting *FG* at *O*; this determines the centre of the circle (Locus 8).

Discussion. The problem is impossible if the distance from *A* to *FG* is $>$ *MP*. If this distance $=$ *MP*, there is one solution; if it is $<$ *MP*, there are two solutions.

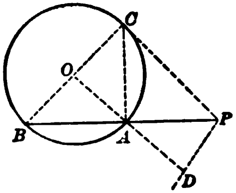
4. Construct a circle which shall touch a given straight line at a given point, and also touch a given circle.



Analysis. Suppose the problem solved, *AB* the given straight line, *C* the given point, *O* the centre of the given circle, *R* the centre of the required circle. One locus for *R* is the straight line *CD* ⊥ to *AB* and erected at *C* (Locus 10). If *DC* is produced to *E*, making *CE* $=$ the radius of the given circle, then $RE = RO$; therefore, another locus for *R* is the right bisector of the line *OE* (Locus 3).

Discussion. The required circle obtained by this analysis touches the given circle *externally*. Another circle answering all the conditions, but touching the given circle *internally*, is obtained by laying off a length equal to *CE* from *C* towards *D*, and proceeding otherwise just as before.

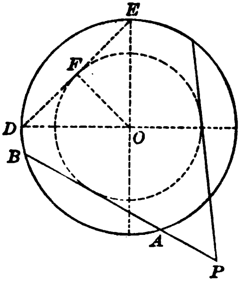
5. Draw through a given point without a given circle a secant so that the intercepted chord shall be equal to the part of the secant which lies without the circle.



Analysis. Let P be the given point, O the centre of the given circle, PAB the secant required. Draw $AC \perp$ to PB, meeting the circumference at C. AC is the right bisector of PB; therefore, $CP = CB$. Moreover, CB is a diameter of the circle (191). Hence C can be found (Locus 1); and then B; and then PAB can be drawn.

Another analysis is effected by producing OA by its own length to D and joining PD. $\triangle PAD = \triangle OAB$; therefore, $PD = OB$; whence D can be determined by a double application of Locus 1; and then by drawing OD, the point A is found.

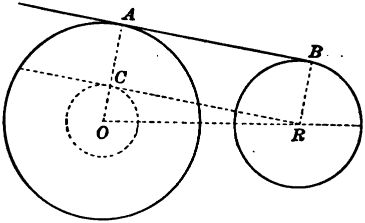
6. Draw through a given point without a given circle a secant so that the intercepted chord shall subtend one-fourth part of the circumference.



Analysis. Let P be the given point, O the centre of the given circle, PAB the secant required. Describe a circle with centre O which shall touch PAB. All chords of the larger circle which are tangent to the smaller circle will be equal to AB (178), and subtend an arc of a quadrant (172). If, then, we draw two diameters of the circle $\perp$ to each other, and draw DE, and then draw $OF \perp$ to DE, we determine the smaller circle; and PAB is simply a tangent to this auxiliary circle.

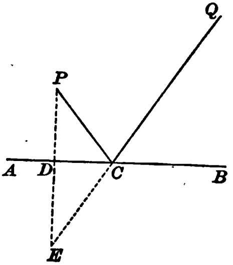
7. Draw a common tangent to two given circles.

Analysis. Let O, R be the centres of the circles, AB the tangent required. The $\angle$s at A and B are right $\angle$s, and if RC is drawn $\perp$ to OA, meeting OA at C, the figure $ABRC$ will be a rectangle. The length of OC is therefore known, since $AC = BR$ and $OC = OA - AC$. It is also evident that RC is tangent to a circle with centre O and radius OC. By describing this auxiliary circle and drawing RC, the point C is determined, and the remainder of the construction is obvious.

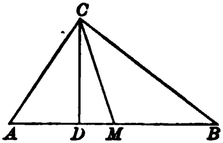


8. Having given a straight line AB, and two points P, Q on the same side of AB, find a point C in AB such that the lines PC, QC shall make equal angles with AB.

Analysis. If we produce QC as shown in the figure, we know that the $\angle$s ACE, QCB are equal; and if we draw $PD \perp$ to AB, and produce it to meet QC at E, we know that the $\triangle$s PDC, EDC are equal; therefore $DE = PD$. Hence E is determined; and by joining E to Q, the point C is found.

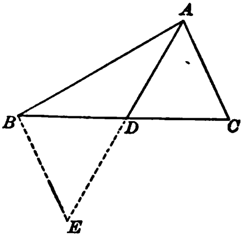


9. Construct a triangle, having given one side, and the altitude and median drawn from the vertex of the angle opposite to the given side.



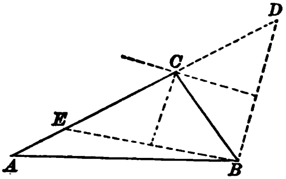
Analysis. Given the lengths of *AB*, *CD*, and *CM*. △ *CDM* is a right △, which is determined by the given parts, and can be constructed; then, since *AM* = *BM*, and the length of *AB* is given, the points *A*, *B* are easily found. By joining *AC*, *BC* the △ is constructed.

10. Construct a triangle, having given the angles and one median.



Analysis. Suppose the problem solved, produce the given median *AD* by its own length to *E* and join *BE*. It is easy to show that the ⚠ *ADC*, *BDE* are equal; whence it follows that *BE* is ∥ to *AC*, and that ∠ *EBC* is the supplement of ∠ *BAC*. One locus for the point *B* is an arc of a segment capable of containing an angle equal to the supplement of the given ∠ *BAC*, and described upon *AE*; another locus for *B* is an arc of a segment capable of containing an angle equal to the given ∠ *ABC*, and described upon *AD*. The point *B* having been determined, the rest of the construction is obvious.

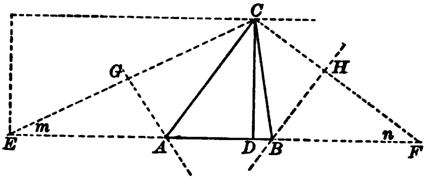
11. Construct a triangle, having given one side, an angle adjacent to this side, and the sum or difference of the other two sides.



Analysis. *Case* 1. Given AB, $\angle BAC$, $AC+CB$. If we produce AC to D, making $CD=CB$, we obtain a $\triangle ABD$, which we can construct, for we know one of its angles and the two including sides. Also, since the $\triangle BCD$ is isosceles, the right bisector of BD will pass through the vertex C and determine this point.

Case 2. Given AB, $\angle BAC$, $AC-CB$. Upon CA take $CE=CB$, and join BE. $\triangle ABE$ can be constructed since we know two sides and the included angle. $\triangle BEC$ is isosceles, and the right bisector of BE will determine the point C.

12. Construct a triangle, having given the perimeter, one angle, and the altitude drawn from the vertex of this angle.



Analysis. Let $\triangle ABC$ be the triangle required, $\angle ACB$ the given angle, CD the given altitude. Produce AB both ways, making $AE=AC$, $BF=BC$; then EF is equal to the given perimeter.

Join *CE*, *CF*; and let m, n denote the angles at *E*, *F* respectively, in the isosceles $\triangle$ *ACE*, *BCF*. In the $\triangle$ *ECF*,

$$2(m+n)+\angle ACB=180° \quad . \ . \ . \ (136)$$

whence $$m+n=90°-\tfrac{1}{2}\angle ACB \quad . \ (112, 114)$$

Now $$\angle ECF=m+n+\angle ACB$$

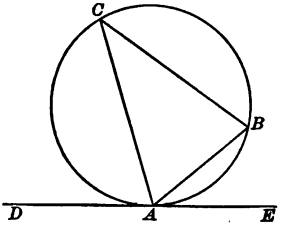
Therefore, by substitution :

$$\angle ECF=90°+\tfrac{1}{2}\angle ACB.$$

This angle is therefore known, and the $\triangle$ *ECF* can be constructed. For one locus for the point *C* is the arc of a segment capable of containing the $\angle$ *ECF* described upon *EF*; another locus is a parallel to *EF* drawn at the distance *CD*.

The points *A* and *B* are found by drawing the right bisectors of *EC* and *FC*.

13. Construct a triangle, having given the radius of the circumscribed circle, and the angles.

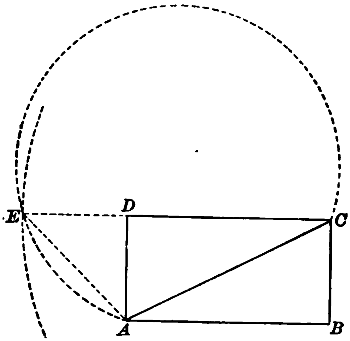


Analysis. Suppose the problem solved, and $\triangle ABC$ the triangle required. Through *A* draw the tangent *DAE*. $\angle DAC=\angle B$, $\angle EAB=\angle C$ (Propositions 190, 193).

Hence, in order to make the construction, describe a circle with the given radius, draw any tangent, and then draw from the point of contact chords making with the tangent angles equal respectively to two of the given angles.

14. Construct a rectangle, having given the perimeter and the diagonal.

Analysis. Suppose the problem solved, *ABCD* the required rectangle, *AC* the given diagonal. Produce *CD* to *E*, making $DE = DA$, and join *AE*. *CE* is equal to half the perimeter, and is therefore known. *AC* is given, and $\angle E = 45°$. Therefore the $\triangle ACE$ can be constructed.



Construction. Draw a line equal to *AC*, the given diagonal; upon *AC* construct a segment capable of containing the angle 45°; with *C* as centre and a radius equal to half the perimeter describe an arc cutting the arc of the segment at *E*; join *CE*; draw *AD* ⊥ to *CE*; draw *AB* ∥ to *CD* and *BC* ∥ to *AD*.

NOTE. — A classification of problems with hints for solution, similar to that already given of theorems, might be made, but it would occupy considerable space if properly done. The aim and scope of the present work does not call for it.

See Wentworth and Hill's "Exercise Manual in Geometry."

CHAPTER IV.

Examination Papers.

Book I. — The Straight Line.

1.

Propositions 126, 130, 136, 140.

1. Two parallels are cut by a transversal so that one of the angles formed is 30°. Draw a figure and write in it the values of all the angles formed.

2. One angle of a triangle is equal to 36°. Find the other angles if one of them is twice as large as the other.

3. Prove that the bisectors of two adjacent angles are perpendicular to each other.

2.

Propositions 132, 133, 135, 145.

1. One angle of a triangle is twice another angle and three times the third angle. Find the value of each angle.

2. Prove that the bisector of the angle at the vertex of an isosceles triangle bisects the base.

3. Prove that the sum of the distances of any point within a triangle from the three vertices is less than the perimeter of the triangle (135).

3.

Propositions 143, 146, 147, 151.

1. If the angle at the vertex of an isosceles triangle is 50°, find the exterior angle formed by producing the base.

2. Prove that the bisectors of the base angles of an isosceles triangle are equal.

3. ABC and ABD are two isosceles triangles, standing on the same base AB. Prove that the straight line CD bisects the angles at C and D, and is perpendicular to AB.

4.

Propositions 131, 144, 149, 152.

1. The angle at the vertex of an isosceles triangle is equal to 40°. Find the angle formed by the bisectors of the base angles.

2. Prove that if the bisector of an angle of a triangle bisects the opposite side, the triangle is isosceles.

3. In a triangle ABC the bisector of the angle A meets BC at D. Prove that AB is greater than BD, and AC is greater than CD (137, 152).

5.

Propositions 154, 155, 157.

1. One angle of a parallelogram is equal to 60°. Find the other three angles.

2. Prove that the diagonals of a rectangle are equal.

3. Prove that the two legs of an isosceles trapezoid make equal angles with each of the bases.

(*Hint:* Drop perpendiculars from the ends of the shorter base to the other base. See 127, 154, 157.)

6.

Propositions 153, 165, 166.

1. Prove that the diagonals of a rhombus are perpendicular to each other.

2. Find the value of each angle of an equiangular decagon.

3. Prove that the lines which join the middle points, taken in order, of a quadrilateral form a parallelogram.

(*Hints:* Draw the diagonals, and apply 165.)

7.

Propositions 156, 164.

1. How many degrees in the angle formed by the bisectors of the acute angles of an isosceles right triangle?

2. Find each angle of an equiangular dodecagon.

3. Prove that the bisectors of the angles of a triangle meet in one point (155, 156).

4. Prove that the middle point of the hypotenuse of a right triangle is equidistant from the three vertices.

(*Hint:* Draw a perpendicular from the middle point to one leg. 127, 164, 145.)

8.

Propositions 161, 163.

1. Prove that two trapezoids are equal if their sides taken in order are equal each to each.

(*Hint:* Divide each figure into a parallelogram and a triangle.)

2. Prove the converse of Proposition 160.

3. In a right triangle ABC, the angle A is equal to 30°, and the angle B is equal to 90°. Prove that BC is equal to $\frac{1}{2}AC$.

(*Hint:* Produce CB to D, making BD equal to CB, and join AD.)

4. Prove that the lines joining the middle points of the sides of a triangle divide the triangle into four equal triangles.

9.

Proposition 151.

1. What angle is formed by the bisectors of two angles of an equilateral triangle?

2. Prove that the perpendiculars erected at the middle points of the sides of a triangle meet in one point.

3. *ABCD* is a square; *AB* is produced by its own length to *E*, and *AD* by its own length to *F*. Prove that *E*, *C*, *F*, are in the same straight line.

(*Hint:* Show that the sum of the angles at *C* is 180°.)

4. Prove that the bisectors of the angles of a parallelogram enclose a rectangle.

5. *ABCD* is a parallelogram; *P*, *Q* the middle points of *AB*, *CD*; prove that *AQ* and *CP* trisect the diagonal *BD* (161, 164).

10.

Proposition 152.

1. Prove that every straight line through the centre of a parallelogram divides it into two equal figures.

2. Prove that if one leg of a right triangle is equal to half the hypotenuse, the angle opposite this leg is equal to 30°.

3. *ABCD* is a square; *BC* is produced to any point *E*. Prove that *AE* is greater than *AC*.

4. Prove that the line joining the middle points of the legs of a trapezoid is parallel to the bases.

5. Prove that if the diagonals of a trapezoid are equal, the trapezoid is isosceles.

BOOK II. — THE CIRCLE.

11.

Propositions 169, 170, 174, 175.

1. How many circles passing through two given points can be described? Upon what line do their centres lie? What is the radius of the smallest of the circles?

2. Prove that the portions of a straight line intercepted between the circumferences of two concentric circles are equal (174).

3. Prove that the radius which bisects an arc bisects also the chord of that arc, and is perpendicular to the chord.

12.

Propositions 172, 177, 181.

1. What is the locus of the centres of all circles that have a given radius, and touch a given straight line?

2. Prove that in two concentric circles all chords of the greater which touch the smaller are equal.

3. Prove that in the same circle parallel lines intercept equal arcs (174).

4. State and prove the converse of No. 3.

13.

Propositions 178, 180, 184.

1. A chord of fixed length slides along the circumference. What is the locus of its middle point?

2. Prove that of all chords which can be drawn through a point within a circle, that is the shortest which is perpendicular to the diameter passing through the point.
(*Hint:* Draw any other chord, and apply 152, 180.)

3. Prove that the sum of two opposite sides of a circumscribed quadrilateral is equal to the sum of the other two sides (184).

14.

Propositions 182, 190.

1. An arc of a segment contains 200°. What is the value of every angle inscribed in the segment?

2. Prove that the opposite angles of an inscribed quadrilateral are supplementary.

3. Prove that the measure of the angle formed by two intersecting chords is half the sum of the arcs intercepted by the sides of the angle and by the sides produced.

4. *AC* and *BD* are two equal intersecting chords. Prove that *ABCD* is an isosceles trapezoid.

15.

Propositions 183, 193.

1. Two tangents drawn from an exterior point form an angle of 60°. How many degrees are there in the arc contained between their points of contact?

2. A right triangle is inscribed in a circle, and one leg is equal to the radius. Prove that its acute angles are 30° and 60°.

3. Prove that a circle described upon one leg of an isosceles triangle as diameter bisects the base (191).

4. Two circles touch externally at *P*, and secants through *P* meet one of the circles in *A*, *C*, the other circle in *B*, *D*. Prove that *AC* is parallel to *BD*.

(*Hint:* Draw at *P* a common tangent. 190, 193.)

16.

Propositions 200, 201.

1. How many degrees in the arc between two parallel chords of which one is a diameter and the other equal to the radius?

2. Prove that the measure of the angle formed by two secants drawn from an exterior point is half the difference of the intercepted arcs.

3. Prove that a circle described on the hypotenuse of a right triangle as diameter will pass through the vertex of the right angle.

(Use the indirect method of proof with 191.)

4. Prove that the line joining the vertex of the right angle of a right triangle to the centre of the square constructed upon the hypotenuse bisects the right angle.

(*Hint:* Describe a circle upon the hypotenuse as diameter.)

17.

Propositions 190, 196, 199.

1. The bisectors of the angles of an inscribed triangle *ABC* meet the circumference in *D*, *E*, *F* respectively; the angle *A* is equal to 70°, the angle *B* is equal to 50°. Find the angles of the triangle *DEF*.

2. From the middle point of an arc *AB*, a perpendicular is drawn to the chord *AB*. Through *A* a tangent is drawn, and a perpendicular to the tangent is erected at *A*. Prove that these perpendiculars are equal.

3. An equilateral triangle *ABC* is inscribed in a circle, and any point *P* in the arc *AB* is joined to *A*, *B*, *C*. Prove that $PA + PB$ is equal to *PC*.

(*Hints:* Upon *PC* take *PD* equal to *PA* and join *AD*.)

18.

Propositions 184, 206, 208.

1. Two tangents drawn from an exterior point to a circle are perpendicular to each other. The radius of the circle is equal to 4 inches. How long is each tangent?

2. Upon one leg of a right triangle as diameter a circle is described. Prove that a tangent drawn through the point where this circle cuts the hypotenuse will bisect the other leg.

3. Prove that a circle can be circumscribed about a rectangle.

(*Hint:* Show that a circle passing through three vertices must pass through the fourth.)

19.

Propositions 210, 211.

1. What is the locus of the centres of all circles that touch a given straight line at a given point?

2. Prove that a circle can be circumscribed about every isosceles trapezoid.
(See hint in Paper 18, No. 3.)

3. Prove that the altitude of an equilateral triangle is equal to three times the radius of the inscribed circle.

4. Construct a circle which shall pass through a given point and touch a given straight line at a given point.

20.

Propositions 209, 212.

1. What is the locus of the middle points of all chords drawn through a fixed point in a circumference? (175, 191.)

2. A common tangent to two circles which touch each other externally at A, touches the circles at B and C respectively. Prove that AB is perpendicular to AC.
(*Hint:* Draw a tangent through A. 184, 147, 136.)

3. Construct a square, having given the diagonal.

4. Construct a triangle, having given the base, the altitude and the angle at the vertex (212).

BOOKS I. AND II. — REVIEW.

21.

Propositions 146, 152.

1. Prove that the bisectors of the base angles of an isosceles triangle are equal.

2. Prove that a point within the sides of an angle, but not on the bisector of the angle, is unequally distant from the sides (155, 152).

3. Prove that the radius perpendicular to one side of an inscribed equilateral triangle is bisected by that side.

4. What is the locus of the centres of all circles that touch a given circle at a given point?

Construct a circle that shall pass through a given point and touch a given circle at a given point.

22.

Propositions 154, 173, 209.

1. Prove that the perpendiculars erected at the middle points of the sides of a triangle meet in one point.

2. Prove that the altitudes of a triangle meet in one point.

(*Hint:* Form a new triangle by drawing lines through the vertices parallel to the opposite sides, and make use of No. 1.)

3. Two circles touch externally at A, and a straight line through A meets the circles again in B and C. Prove that the tangents at B and C are parallel.

4. Construct an equilateral triangle, having given its perimeter.

23.

Propositions 137, 164, 186.

1. Prove that the lines joining the vertices *B* and *D* of the parallelogram *ABCD* to the middle points respectively of *CD* and *AB* trisect the diagonal *AC* (161, 164).

2. Give a direct proof that an angle inscribed in a semi-circle is equal to a right angle.
(*Hint:* Join the vertex of the angle to the centre, and apply 147, 136.)

3. Construct a right triangle, having given the hypotenuse and the altitude upon the hypotenuse.

4. Inscribe a circle in a given sector (155, 186).

24.

Propositions 140, 157.

1. Prove that the straight lines joining the middle points of the sides of a rectangle, taken in order, form a rhombus.

2. Two tangents drawn from a point to a circle form an angle of 60°. How many degrees in the angle between one of these tangents and the chord joining the points of contact ?

3. Two circles touch internally at *A*, and a chord *BC* of the greater circle touches the smaller at *D*. Prove that *AD* bisects the angle *BAC*.
(*Hint:* Draw a tangent through *A*. 193.)

4. Construct an equilateral triangle, having given its altitude.

25.

Propositions 151, 158, 211.

1. A triangle *ABC* is inscribed in a circle. The angle *B* is equal to 50°, and the angle *C* is equal to 60°. What angle does a tangent at *A* make with *BC* produced to meet it?

2. Prove that two trapezoids are equal if their sides are equal, taken in order, pair by pair.

(*Hint:* Divide each trapezoid into a parallelogram and a triangle.)

3. Prove that a circle can always be inscribed in a rhombus.

4. Construct a trapezoid, having given its sides.

26.

Propositions 149, 154, 163.

1. If one of the acute angles of a right triangle is double the other, prove that one of the legs is equal to half the hypotenuse.

2. Two equal circles intersect in *A* and *B*. A straight line through *A* meets one circle in *P*, the other in *Q*. Prove that *BP* is equal to *BQ*.

3. Construct a rectangle, having given one side and a diagonal.

4. Construct a circle which shall have a given radius and touch two given intersecting straight lines.

27.

Propositions 147, 153, 190.

1. What is the measure of the angle formed by two intersecting chords?

2. Prove that the middle point of the hypotenuse of a right triangle is equidistant from the three vertices.

3. The altitude of a certain right triangle, taking the hypotenuse as base, is equal to half the hypotenuse. Find the values of the acute angles.

4. Within a given circle draw a chord of given length parallel to a given straight line (177, 201).

28.

Propositions 152, 161, 211.

1. Prove that the diagonals of a rhombus are perpendicular to each other.

2. ABC is an isosceles triangle inscribed in a circle. The bisectors of the base angles B and C meet the circumference in D and E respectively. Prove that $BCDAE$ has four of its sides equal. What relation must exist among the angles of the triangle ABC in order that all the sides may be equal?

3. Construct a rhombus, having given its diagonals.

4. Divide a straight line into four equal parts.

29.

Propositions 154, 173, 181.

1. *ABC* is a right triangle. The angle *B* is equal to 90°; *AD* bisects the angle *A* and meets *BC* at *D*. Prove that *DC* is greater than *BD* (see page 26).

2. Prove that a straight line bisecting one leg of a trapezoid and parallel to the bases bisects the other leg.

3. Two circles intersect at *P* and *Q*, and two straight lines, *APB* and *CQD*, are drawn, meeting the circumferences at *A* and *B*, and *C* and *D* respectively. Prove that *AC* is parallel to *BD*.

(*Hint:* Draw the common chord, and consider the angles of the quadrilaterals thus formed.)

4. Place a line of given length between the sides of a given angle so as to form an isosceles triangle.

(*Hint:* Suppose the problem solved, and draw the bisector of the angle.)

30.

Propositions 161, 181, 192.

1. Upon the diagonal *BD* of the parallelogram *ABCD*, *P* and *Q* are taken so that *BP* is equal to *DQ*. Prove that *APCQ* is a parallelogram.

2. What is the measure of the angle formed by two secants drawn from a point without a circle?

3. A circle is inscribed in a rhombus. Prove that the lines which join the points of contact, taken in order, form a rectangle.

4. Draw a common tangent to two given circles.

31.

Proposition 165.

1. In a triangle ABC, angle A is equal to 50°, angle B is equal to 70°. Find the angle which the bisector of angle C makes with the right bisector of the side AB. (*Hint:* Circumscribe a circle about the triangle.)

2. Prove that the diagonals of a rhombus are unequal.

3. Find the locus of the centres of circumferences having a given radius and bisecting a given circumference.

4. Construct a triangle, having given the middle points of its three sides.

5. Construct a circle which shall touch a given circle, have its centre in a given straight line, and cut this line in a given point (see page 54, Ex. 4).

32.

Proposition 192.

1. Prove that the bisectors of the angles of a quadrilateral form a new quadrilateral whose opposite angles are supplementary.

2. Two circles touch at P. APB, CPD are any two straight lines through P, meeting one circle in A, C, the other in B, D. Prove that AC is parallel to BD.

3. Find the locus of all circumferences, having a given radius, and cutting a given circumference at right angles.

4. Construct a triangle, having given the base, the angle opposite the base, and the median drawn to the base (see page 53, Ex. 2).

5. Place between the sides of a given angle a straight line of given length, so that it shall be parallel to a given straight line.

33.

Proposition 188.

1. Prove that the opposite angles of an inscribed quadrilateral are supplementary.

State the converse theorem.

2. Prove that the altitudes *AD, BE, CF,* of the triangle *ABC* bisect the angles of the triangle *DEF.*

(*Hint:* First show by means of the converse of No. 1 that circles can be circumscribed about the figures *AFOE, EODC, AFDC.*)

3. Prove that the diagonals of an isosceles trapezoid are equal.

4. Find a point in a given straight line such that a tangent drawn from the point to a given circle shall have a given length (Locus 23).

5. Construct a right triangle, having given the hypotenuse and the sum of the legs (see page 58, Ex. 11).

34.

Proposition 182.

1. Prove that the difference of the angles at the base of a triangle is equal to the difference of the angles at the vertex made by drawing the altitude of the triangle.

2. Prove that the radius of a circle inscribed in an equilateral triangle is equal to one-third of the altitude of the triangle.

3. Two circles touch internally at *P.* A chord *AB* of the larger circle touches the smaller circle at *C.* Prove that *PC* bisects the angle *APB.*

4. Construct a triangle, having given the angles and one altitude.

5. Through a given point in a given circle draw a chord having a given length (see page 55, Ex. 6).

35.

Proposition 179.

1. In a triangle *ABC*, the angle *A* is equal to 70°, and the angle *B* is equal to 40°. *BC* is produced to *D*, making *CD* equal to *CA*, and *A* is joined to *C*. Find the value of the angle *BAD*.

2. Prove that the lines joining the middle points of the sides of an isosceles trapezoid, taken in order, enclose a rhombus or a square. When will the figure be a square ?

3. Prove that the perimeter of a triangle is greater than the sum of its medians (apply Ex. 3, page 30).

4. Construct an equilateral triangle, having given the sum of the altitude and one side (see page 58, Ex. 11).

36.

Proposition 176.

1. Prove that any straight line passing through the centre of a parallelogram divides the perimeter into two equal parts.

2. From a point *P* outside a circle a line *PAB* is drawn through the centre of the circle, cutting the circumference in *A* and *B*; also a secant *PCD*, so that *PC* is equal to the radius; *D* is joined to *O*, the centre of the circle. Prove that the angle *BPD* is equal to one-third of the angle *BOD*.

3. Prove that a circle can be circumscribed about an isosceles trapezoid.

4. Construct an isosceles right triangle, having given its perimeter (see page 58, Ex. 12).

5. Through a fixed point within a circle chords are drawn. Find the locus of their middle points.

37.

Proposition 180.

1. Prove that the perpendiculars erected at the middle points of the sides of a triangle are concurrent (meet in one point).

2. Prove that the three altitudes of a triangle are concurrent.

(*Hint:* Draw through the vertices lines parallel to the opposite sides and try to reduce the theorem to No. 1.)

3. AB is a diameter of a circle, OM any radius, and MC a perpendicular from M to AB. Upon OM a length OP equal to MC is laid off. Find the locus of P.

4. Construct a triangle, having given two sides and the median included between them (see page 57, Ex. 10).

5. Draw through a given point P a line forming an isosceles triangle with the sides of a given angle.

38.

Proposition 160.

1. In a triangle ABC, BC is produced to D, making CD equal to CA, and upon BD a length BE equal to BA is laid off, and A is joined to E. Prove that AE bisects the angle BAD.

2. Prove that if perpendiculars are drawn from the vertices of a parallelogram to any straight line the sum of the two drawn from two opposite vertices is equal to the sum of the two others. (*Hint:* Draw the diagonals.)

3. Construct a triangle, given one angle, one side of this angle, and the median drawn from this angle (see page 57, Ex. 10).

4. Draw a line in a triangle ABC meeting AC at D, BC at E, so that $CD = DE = EB$.

39.

Proposition 161.

1. Two circles touch externally at *A* and their common tangent touches them at *B* and *C*. Prove that *AB* is perpendicular to *AC*.

2. Prove that the line joining the middle points of the legs of a trapezoid is parallel to the parallel sides and equal to half their sum.

3. Diameters are drawn through one point of intersection of two circles. Prove that the straight line joining their ends passes through the other point of intersection.

4. Construct an isosceles triangle, given the perimeter and the angle at the vertex.

5. Draw through one of three given points a straight line equidistant from the other two.

40.

Proposition 149.

1. Lines parallel to one another and equal in length are drawn through the vertices of a polygon. Prove that the polygon formed by joining their ends is equal to the given polygon.

2. What measures the angle formed by two intersecting chords?

3. A secant *PAB* and a tangent *PT* are drawn to a circle from an exterior point *P*. The bisector of the angle *ATB* meets *AB* at *C*. Prove that *PC* is equal to *PT*.

4. Construct a square, given the sum of the diagonal and one side.

5. Through one point of intersection of two given circles draw a straight line so that the two intercepted chords shall be equal.

Book III. — Similar Figures.

41.

Propositions 218, 219, 236.

1. Prove that if any two straight lines are cut by three parallels they are cut proportionately.

2. From a point *P* in the common base of two triangles *ACB*, *ADB*, straight lines are drawn parallel to *AC*, *AD*, meeting *BC*, *BD* at *E* and *F* respectively. Prove that *EF* is parallel to *CD*.

3. The sides of a triangle are 9, 12, 15 ; find the segments of the sides made by the bisectors of the angles (220).

42.

Propositions 220, 221, 223.

1. The side *BC* of a triangle *ABC* is bisected at *D*, and the bisectors of the angles *ADB*, *ADC* meet *AB*, *AC* at *E* and *F* respectively. Prove that *EF* is parallel to *BC*.

2. The bases of a trapezoid are 8 and 12, and the altitude is 6. Find the altitudes of the two triangles formed by producing the bases until they intersect.

3. Construct a triangle, having given the base, the vertical angle, and the ratio of the other two sides (192, 220).

43.

Propositions 224, 225, 229.

1. Prove that the diagonals of a trapezoid cut each other in the same ratio.

2. *ABCD* is a quadrilateral. Prove that if the bisectors of the angles *A* and *C* meet on the diagonal *BD*, the bisectors of the angles *B* and *D* will meet on the diagonal *AC* (220).

3. How far apart are the opposite corners of a floor 12 feet by 16 feet?

4. Produce a straight line *AB* to a point *C* so that $AB : AC = 4 : 7$.

44.

Propositions 221, 230, 239.

1. Prove that the sum of the squares of the segments formed by two perpendicular chords is equal to the square of the diameter of the circle.

2. From one angle *A* of a parallelogram a straight line is drawn cutting the diagonal *BD* in *E*, the side *DC* in *P*, and the side *BC* produced in *Q*. Prove that *AE* is a mean proportional between *EP* and *EQ*.
(*Hint:* Compare triangles *AED*, *BEQ*, and triangles *AEB*, *DEP*.)

3. The radii of two circles are 8 inches and 3 inches, and the distance between their centres is 15 inches. Find the length of their common exterior tangent.

45.

Propositions 219, 233, 237.

1. Two circles touch externally, and through the point of contact three secants are drawn. Prove that by joining the points where these secants meet the circumferences two similar triangles are formed.

2. From a point *P*, tangents *PA*, *PB* are drawn to a circle whose centre is *C*, and *AD* is drawn perpendicular to *BC* produced. Prove that *AD* is the fourth proportional to *PB*, *BC*, and *BD*.

3. If one side of an equilateral triangle is equal to a, find the altitude.

46.

Propositions 235, 242.

1. Through a point *P* exterior to a circle, a tangent *PA* and a secant *PBC* are drawn. If *PB* is equal to 5 inches, and *BC* is equal to 4 inches, find *PA*.

2. Find the locus of points which divide all straight lines drawn from a given point to a given straight line in a given ratio.

3. Two circles touch externally at *P*, and a common tangent touches them at *A* and *B*. Prove that *AB* is a mean proportional between the diameters of the circles.

4. Find in one side of a given triangle a point, the distances of which from the other two sides shall have a given ratio.

47.

Propositions 222, 231.

1. The radius of a circle is equal to 10 inches. Through a point 6 inches from the centre a diameter is drawn, and also a chord perpendicular to the diameter. Find the length of this chord and the distances from one end of the chord to the ends of the diameter.

2. Prove that the product of the legs of a right triangle is equal to the product of the hypotenuse and the altitude upon the hypotenuse.

3. In the sides AB, AC of a triangle ABC two points D and E are taken such that BD is equal to CE. If DE and BC produced meet at F, prove that $AB : AC = EF : DF$.

(*Hint:* Draw EG parallel to AB.)

4. Give a geometric construction for cutting off three-sevenths of a given straight line.

48.

Propositions 226, 234.

1. Through a point P, 2 inches from the centre of a circle, a chord APB is drawn. The product $AP \times PB = 21$. Find the radius of the circle (233).

2. Prove that the radius of a circle is a mean proportional between the segments of any tangent made by the point of contact and a pair of parallel tangents.

3. At the ends of a diameter of a circle tangents are drawn meeting a tangent drawn through P in the circumference at Q and R. Prove that the product $PQ \times PR$ is constant for all positions of P.

4. Draw through a given point P on the arc subtended by a chord AB a chord which shall be bisected by AB.

49.

Propositions 227, 232.

1. Through a point 10 inches from the centre of a circle whose radius is 6 inches, tangents are drawn. Find their lengths and also the length of the chord joining the points of contact.

2. Prove that the tangents to two intersecting circles drawn from any point in their common chord produced are equal (235).

3. AB is a diameter of a circle, and the straight line ACD cuts the circle in C and the tangent at B in D. Prove that the product $AC \times AD$ is constant.

4. Inscribe in a given circle a triangle similar to a given triangle.

50.

Propositions 228, 240.

1. If the altitude of an equilateral triangle is equal to h, find the length of one side.

2. A is the vertex of an isosceles triangle ABC inscribed in a circle, and ADE is a straight line which cuts the base in D and the circle in E. Prove that $AE \times AD = \overline{AB}^2$.

3. Two circles touch internally. Prove that all chords of the larger circle drawn through the point of contact are divided proportionally by the circumference of the smaller circle.

4. Construct a circle which shall pass through two given points and touch a given straight line (235).

Book IV. — Equivalent Figures.

51.

Propositions 243, 246, 248.

1. Find the area of an isosceles triangle if the base is equal to 36 feet and one leg is equal to 30 feet.

2. Two triangles have equal altitudes and their bases are 20 feet and 30 feet. Find the base of a triangle equivalent to their sum, and having an altitude equal to one-fourth of their common altitude.

3. Prove that two parallelograms which have two adjacent sides equal and the included angles supplementary are equivalent.

4. Prove that the sum of the areas of two opposite triangles formed by joining any point within a parallelogram to the four vertices is equal to half the area of the parallelogram.

52.

Propositions 244, 250, 258.

1. The bases of a trapezoid are a, b. The altitude of the triangle formed by producing the legs of the trapezoid till they meet is h. Find the area of the trapezoid.

2. Prove that the area of a rhombus is equal to half the product of its diagonals.

3. Prove that the area of a trapezoid is equal to the product of one leg and the distance from this leg to the middle point of the other leg.

(*Hint:* Draw a line through the middle point parallel to the opposite leg.)

4. Bisect a given parallelogram by a line through a given point within the parallelogram.

53.

Propositions 251, 252, 255.

1. The bases of a trapezoid are 32 feet and 12 feet. Each of the legs is 10 feet long. Find the area of the trapezoid.

2. *ABCD* is a trapezoid, *AB* is parallel to *CD*, and the diagonals *AC*, *BD* intersect at *O*. Prove that triangle *AOD* is equivalent to triangle *BOC* (222, 252).

3. Prove that the line joining the middle points of two adjacent sides of a parallelogram cuts off a triangle equivalent to one-eighth of the parallelogram.

4. Construct an isosceles triangle equivalent to a given triangle, and having its vertical angle equal to one of the angles of the given triangle (252, 239).

54.

Propositions 253, 257.

1. The base of a triangle is equal to 32 feet, and the altitude is equal to 20 feet. What is the area of the triangle formed by a straight line parallel to the base, and distant 5 feet from the vertex ?

2. Prove that the areas of two similar triangles are to each other as the squares of the radii of their inscribed circles.

3. Prove that the area of a circumscribed polygon is equal to the product of half its perimeter and the radius of the circle.

4. Transform a given triangle into an equivalent isosceles triangle (249).

55.

Propositions 254, 255.

1. Upon the sides of a right triangle, taken as homologous sides, three similar polygons are constructed. Prove that the sum of the areas of the polygons constructed upon the two legs is equal to the area of the polygon constructed upon the hypotenuse.

2. Find the area of the equilateral triangle equivalent to the sum of the areas of two equilateral triangles whose sides are 3 feet and 4 feet respectively.

3. The side AB of a triangle ABC is produced to D, and D is joined to C; BE is drawn parallel to CD, and meeting AC at E. Prove that the triangle ADE is equivalent to the triangle ABC.

4. Transform a given triangle into an equivalent isosceles triangle, having a given length for its base. (*Hint:* Apply No. 3.)

56.

Propositions 253, 260.

1. Prove that two similar triangles are to each other as the squares of the medians drawn from equal angles.

2. If the side of one equilateral triangle is equal to the altitude of another, compare their areas.

3. Construct a square equivalent to one-half of a given square.

4. Divide a triangle into two equivalent parts by a line parallel to one side of the triangle.

57.

Propositions 251, 259.

1. What part of a parallelogram *ABCD* is cut off by a line joining the middle point of *AD* to a point in *BC* two-thirds of the distance from *B* to *C*?

2. From the ends *A*, *B* of the hypotenuse of a right triangle *ABC*, lines *AE*, *BD* are drawn perpendicular to *AB*, and meeting *BC* and *AC* produced in *E* and *D* respectively. Prove that the triangles *ABC*, *DCE* are equal in area (248).

3. Construct a triangle equivalent to the sum of two given triangles (259).

4. Transform a square into an equivalent isosceles triangle.

58.

Propositions 246, 256.

1. Through a point *D* in the side *AB* of a triangle *ABC*, a line *DE* is drawn parallel to *BC*, and so that the figure *BDEC* is three-fourths as large as the triangle *ABC*. If *AB* is 10 inches, find the length of *AD*.

2. *M* is the middle point of the base *AB* of a triangle *ABC*, and *P* any other point in the base. If *MQ* is drawn parallel to *PC*, meeting *AC* in *Q*, and *P* is joined to *Q*, prove that the triangle *APQ* is equal to one-half the triangle *ABC*.
(*Hint:* Join *MC*, 249.)

3. Bisect a triangle by drawing a line through a given point in one of its sides.

4. Construct a square three times as large as a given square.

59.

Propositions 250, 257.

1. The greater base of a trapezoid is equal to 10 feet, the altitude is equal to 4 feet, and the area is equal to 32 square feet. What is the length of a line drawn between the legs parallel to the given base and distant one foot from it?

2. Prove that the equilateral triangle constructed on the hypotenuse of a right triangle is equivalent to the sum of the equilateral triangles upon the legs.

3. Transform a trapezoid into an equivalent square.

4. Find a point within a triangle such that straight lines drawn from it to the three vertices will divide the triangle into three equivalent parts.

(*Hint:* Divide the triangle into three equivalent parts by lines drawn from one vertex. See also 249.)

60.

Propositions 251, 258.

1. If an altitude h of a triangle is increased by m, by how much must the base be diminished in order that the area may be unchanged?

2. Prove that the figure enclosed by straight lines joining the middle points of the sides of a quadrilateral, taken in order, is equivalent to one-half the quadrilateral.

3. Construct a square which shall be to a given square as 4 : 3.

4. Divide a triangle into two equivalent parts by a line perpendicular to one of the sides.

(*Hint:* Draw the altitude parallel to the required line, and apply 251.)

Book V. — Regular Figures.

61.

Propositions 261, 271.

1. Find the area of a regular hexagon, one side of which is *a* feet long.

2. The span (chord) of a bridge in the form of a circular arc is 96 feet, and its height above the piers is 12 feet. Find the radius of the circle.

3. Upon the six sides of a regular hexagon squares are constructed outwardly. Prove that the exterior vertices of these squares are the vertices of a regular dodecagon.

4. Construct a regular octagon, having given one side.

62.

Propositions 262, 273.

1. The radius of a circle is equal to 14 feet. What is the length of the arc subtended by one side of the inscribed regular decagon?

2. What is the ratio of the area of an inscribed equilateral triangle to that of the inscribed regular hexagon? Prove your answer to be correct.

3. Prove that the radius of an inscribed regular polygon is a mean proportional between the apothem and the radius of the similar circumscribed regular polygon.

4. Divide a given circumference into two parts which shall be to each other as 3 : 7.

63.

Propositions 263, 274.

1. The chord of half an arc is 12 feet, and the radius of the circle is 18 feet. Find the height of the arc.

2. Find the angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle (189, 267).

3. In a regular hexagon all the diagonals are drawn. Prove that another regular hexagon is formed. Find the ratio of the areas of the two hexagons.

4. Construct a regular dodecagon, having given one side.

64.

Propositions 264, 272.

1. What is the ratio of a square inscribed in a circle to the square circumscribed about the same circle? Prove your answer correct.

2. The chord of an arc is equal to 24, and the height of the arc is equal to 9 inches. Find the diameter of the circle.

3. Prove that twice the square of one side of an inscribed equilateral triangle is equal to three times the area of the square inscribed in the same circle.

4. Prove that the diagonals of a regular pentagon form another regular pentagon.

65.

Propositions 265, 267.

1. The radius of a circle is equal to r. Find the area of the inscribed equilateral triangle.

2. The radius of a circle is equal to r. Find the length of the arc subtended by one side of the inscribed regular decagon.

3. Prove that the radius of an inscribed regular polygon is a mean proportional between the apothem and the radius of the similar circumscribed regular polygon.

4. Circumscribe about a given circle a regular octagon.

66.

Propositions 268, 274.

1. Find the area of the largest square that can be made from a circular piece of wood whose radius is equal to 20 inches.

2. Prove that a circular ring is equal in area to a circle whose diameter is a chord of the outer circle and a tangent to the inner circle.

3. The sides ot a regular hexagon are produced till they meet. Prove that by joining the six points of intersection another regular hexagon is formed, and find the ratio of its area to that of the given hexagon.

4. Construct a circle having double the area of a given circle.

67.

Propositions 261, 275.

1. Find the area of the largest circle that can be made from a square piece of wood whose side is 2 feet, 11 inches long.

2. Three equal circles are described, each touching the two others. If the common radius is r, find the area contained between the circles.

3. Prove that the area of an inscribed regular octagon is equal to the product of the diameter of the circle and the side of the inscribed square.

(*Hint:* Consider a quarter of the octagon as the sum of two triangles having as common base a side of the inscribed square.)

4. Construct a circle equivalent to the sum of two given circles.

68.

Propositions 262, 276.

1. The side of an equilateral triangle is equal to a. Find the areas of the inscribed and the circumscribed circles.

2. Upon each side of a square as diameter, semicircles are described within the square forming four leaves. If the side of the square is equal to a, find the sum of the areas of the four leaves.

3. Prove that the area of an inscribed regular dodecagon is equal to three times the square of the radius of the circle (see hint to Paper 67, No. 3).

4. Having given a circle, construct a concentric circle having half the area of the given circle.

69.

Propositions 268, 269.

1. Find the area of a regular hexagon inscribed in a circle whose circumference is equal to 88 feet.

2. Prove that an angle of a regular pentagon is trisected by the straight lines which join it to the opposite vertices.

3. Prove that the area of an inscribed regular hexagon is equal to three-fourths of the area of the similar circumscribed hexagon.

4. Construct a circle equal in area to three-fourths of a given circle.

70.

Proposition 277.

1. *A*, *B*, *C*, *D* are successive points of division of a circumference divided into 10 equal arcs. The radii *OA*, *OB*, *OD*, and also the chords *AB*, *AD*, are drawn, *AD* cutting *OB* at *E*. Prove that two pairs of isosceles triangles are formed, and find the values of their angles.

2. If r is equal to the radius of a circle, and a to the side of the inscribed regular decagon, prove that $a = \frac{r}{2}(\sqrt{5} - 1)$. (Use the figure of No. 1.)

3. Prove that the square of one side of a regular inscribed pentagon is equal to the sum of the squares of the radius of the circle and the side of the inscribed regular decagon.

4. *AOB*, *COD* are perpendicular diameters of a circle. With *E*, the middle point of *OA*, as centre, and *EC* as radius, an arc is described cutting *OB* at *F*. Prove that *OF* is equal to a side of the inscribed regular decagon, and that *CF* is equal to a side of the inscribed regular pentagon.

BOOKS I.–V. — GENERAL REVIEW.

71.

Propositions 151, 181, 262.

1. The hypotenuse of an isosceles right triangle is 8 inches long. Find the area of the triangle formed by joining the middle points of its three sides.

2. Two circles touch externally, and through the point of contact three straight lines are drawn, meeting one of the circles in *A, B, C*; the other circle in *D, E, F.* Prove that the triangle *ABC* is similar to the triangle *DEF.*

3. Construct a rectangle equivalent to a given trapezoid and having a given length for one side.

4. One side of a regular hexagon is 6 inches. Find the area of the hexagon, and the circumference and area of a circle circumscribed about a regular hexagon whose area is four times as great.

72.

Propositions 163, 231, 253.

1. Prove that if a quadrilateral be circumscribed about a circle, the sum of the angles subtended at the centre by any two opposite sides is two right angles.

2. Describe a circle of given radius, having its centre in a given straight line, and touching another given straight line.

3. *ABC* is an isosceles triangle. A perpendicular to one leg *AB*, erected at *A*, meets the base *BC*, produced (if necessary) at *D*. If *E* is the middle point of *BD*, prove that *AB* is a mean proportional between *BC* and *BE.*

4. A regular hexagon having a perimeter of 36 feet is inscribed in a circle. Find the area of that part of the circle which lies without the hexagon.

73.

Propositions 161, 221, 252.

1. AB is any chord of a circle, AC the diameter through A, and AD the perpendicular to the tangent at B. Prove that AB bisects the angle CAD.

2. Describe a circle having a given radius, and touching a given circle and also a given straight line.

3. Upon two sides of any triangle AB, AC, squares $ABDE$, $ACFG$ are described externally to the triangle. Prove that the triangles DAG, FAE are equivalent (252).

4. The perimeter of a regular hexagon is 48 feet. Find the area of a circle inscribed in an equivalent square.

74.

Propositions 165, 212, 223.

1. Draw a straight line so that it shall be equally distant from the three vertices of a given triangle. How many such lines can be drawn.

2. Prove that the two straight lines which join the middle points of the opposite sides of any quadrilateral bisect each other.

3. Prove that the perpendicular from any point of a circumference upon a chord is a mean proportional between the perpendiculars from the same point upon the tangents drawn at the ends of the chord.

4. The area of a regular hexagon inscribed in a circle is equal to $24\sqrt{3}$. Find the area of the circle.

75.

Propositions 180, 233, 257.

1. Draw a straight line through one vertex of a triangle so that it shall be equally distant from the other two vertices.

2. *ABCD* is a parallelogram. From *B* any straight line is drawn cutting the diagonal *AC* at *E*, the side *CD* at *F*, and the side *AD* produced at *G*. Prove that *BE* is a mean proportional between *EF* and *EG*.

(*Hint:* Draw through *E* a line parallel to *AB*.)

3. Prove that if a straight line is divided into any two parts, the square on the whole line is equivalent to the sum of the squares on the two parts plus twice the rectangle contained by the two parts.

4. Find the area of an equilateral triangle inscribed in a circle, the radius of which is 10 inches.

76.

Propositions 153, 235, 248.

1. A chord *AB* 5 inches long is drawn in a circle whose diameter is 13 inches. Find the distance from *B* to the diameter drawn through *A*.

2. Construct a triangle, having given the base, the vertical angle, and the length of the bisector of the vertical angle.

3. Prove that the area of a triangle is equal to one-half the product of its perimeter by the radius of the inscribed circle.

4. Three equal circles are described so that each circle touches the other two. If the radius of each circle is r, find the area of the surface contained between the circles.

77.

Propositions 136, 237, 261.

1. Prove that a perpendicular drawn from one end of the base of a triangle to the bisector of the vertical angle makes with the base an angle equal to half the difference of the angles at the base.

2. *ABC* is an isosceles triangle, and *DE* is drawn parallel to the base *BC*. Prove that a circle can be described through the four points *B*, *C*, *D*, *E*.

3. On a given straight line as base describe an isosceles triangle equivalent to a given triangle.

4. Upon each side of a square as a diameter semicircles are described within the square, forming four leaves. If a side of the square is 10 inches find the sum of the areas of the four leaves.

78.

Propositions 165, 235, 258.

1. Prove that the medians of a triangle meet in a common point, and that this point divides each median in the ratio 2 : 1.

2. The medians *AD*, *BE* of a triangle *ABC* meet at *F*, and *ED* is drawn. Prove that the triangle *DEF* is one-fourth as large as the triangle *ABF*.

3. Describe a circle which shall pass through two given points and touch a given straight line (see 235).

4. If the radius of a circle is r, find the area of the segment subtended by one side of the inscribed equilateral triangle.

79.

Propositions 161, 231, 253.

1. Prove that the bisectors of the angles of a parallelogram enclose a rectangle, and that the diagonals of this rectangle are parallel to the sides of the parallelogram.

2. Prove that if a circle is inscribed in a right triangle the sum of its diameter and the hypotenuse of the triangle is equal to the sum of the legs.

3. Divide a triangle into three equivalent parts by drawing lines parallel to one side.

4. The distance between the stone piers of a bridge is 240 feet. The bridge is in the form of an arc of a circle, and rises 24 feet above the top of the piers. Find the radius of the circle.

80.

Propositions 232, 246.

1. Prove that if the hypotenuse of a right triangle is double one of the legs, the acute angles are 30° and 60°.

2. Two radii OA, OB of a circle are perpendicular to each other. Prove that their right bisectors divide the arc AB into three equal parts (170).

3. Construct a square equivalent to one-third of a given square.

4. Draw lines from one vertex of a parallelogram so as to divide it into three equivalent parts.

5. The apothem of an equilateral triangle is a. Find the circumference of the circumscribed circle.

81.

Propositions 185, 225, 234.

1. *ABC* is a right triangle with equal legs *AB* and *BC*. If *BC* is produced to *D*, making *CD* equal to *AC*, and *AD* is drawn, find the angle *ADB*.

2. If in No. 1 $AC = 4\sqrt{2}$ inches, find the area of the triangle *ABD*.

3. Construct a circle which shall touch a given straight line, and also touch a given circle at a given point.

4. Prove that of all triangles having a given base and a given angle opposite this base, the isosceles triangle has the greatest area.

82.

Propositions 193, 251.

1. Two circles intersect at *A* and *B*. Through *P*, any point on the circumference of one of them, straight lines *PAC*, *PBD* are drawn to cut the other circle at *C* and *D*. Prove that *CD* is parallel to the tangent at *P*.

2. Transform a right triangle into an isosceles triangle, having for its vertical angle one of the acute angles of the right triangle (251).

3. Divide a triangle into two equivalent parts by drawing a straight line from a given point in one of its sides (249).

4. Construct two concentric circles so that all chords of the larger which touch the smaller circle shall be equal to the diameter of the smaller circle.

5. What is the ratio of the areas of the two circles in No. 4?

CHAPTER V.

Entrance Examinations.

HARVARD COLLEGE, *June,* 1890.

1. Prove that if two sides of a triangle are respectively equal to two sides of another, but the included angle in the first triangle is greater than the included angle in the second, the third side of the first triangle is greater than the third side of the second.

If the included angle in the case of the first triangle is twice as great as the included angle of the other, is the third side of the one twice as long as the third side of the other?

2. Prove that through three points not in the same straight line one circumference can be drawn and only one.

A certain equilateral triangle has sides $8\sqrt{3}$ inches long; what is the radius of the circumference circumscribed about this triangle?

3. Prove that an angle which has its vertex within a circumference is measured by half the sum of the two arcs intercepted between its sides when extended in both directions.

The vertices of a quadrilateral inscribed in a circle divide the circumference into arcs which are to each other as 1, 2, 3, and 4. Find the angles between the opposite sides of the quadrilateral.

4. Prove that the diagonals and the line which joins the middle points of the parallel sides of a trapezoid meet in a point.

5. The radius of a certain circle is 9 inches; find the area of that one of all the regular polygons inscribed in it which has the shortest perimeter. How long a perimeter can a regular polygon inscribed in this circle have?

HARVARD COLLEGE, *September*, 1890.

1. Prove that the sum of the lengths of two straight lines, drawn from an outside point to the ends of a straight line, is greater than the sum of the lengths of any other two straight lines similarly drawn but included by them.

2. Prove that the line which joins the middle points of the non-parallel sides of a trapezoid bisects both diagonals.

3. Prove that one, and only one, circumference can be drawn so as to touch any three lines which do not pass through a single point.

A certain equilateral triangle has sides $8\sqrt{3}$ inches long; what is the radius of the circumference inscribed in this triangle?

4. Prove that if the sides of an angle which has its vertex without a circle intercept the circumference, the angle is measured by half the difference of the arcs included between them.

The points of tangency of a quadrilateral, circumscribed about a circle, divide the circumference into arcs, which are to each other as 4, 6, 10, and 16. Find the angles of the quadrilateral.

5. The radius of a certain circle is 9 inches; find the area of that one of all the regular polygons which can be circumscribed about it which has the longest perimeter. How short a perimeter can a regular polygon circumscribed about this circle have?

YALE COLLEGE, *June*, 1890.

1. Prove that the bisectors of the angles of a triangle meet in a point.

2. On a given straight line *AB* construct a segment of a circle containing an angle equal to a given acute angle *C*.

(Show clearly in the figure the method of construction. Ruler and compass must be used. The proof is not required.)

3. Prove that in any triangle the square of the side opposite an acute angle is equivalent to the sum of the squares on the other two sides diminished by twice the product of one of these sides and the projection of the other upon that side.

4. Prove that regular polygons of the same number of sides are similar figures.

5. If the radius of a circle is 5 inches, compute its circumference and its area; also the perimeter, the area, and the apothegm of an inscribed square.

SHEFFIELD SCIENTIFIC SCHOOL, *June*, 1890.

1. Two straight lines which are parallel to a third straight line are parallel to each other.

2. Two sides of a triangle, and the angle opposite one of them, are given. Construct the triangle, giving all the possible solutions.

3. The perimeter of the circumscribed equilateral triangle is double that of the similar inscribed triangle.

4. Similar triangles are as the squares on any two homologous lines.

5. The circumference is the limit of the perimeters of the circumscribed and inscribed similar regular polygons when the number of sides is indefinitely increased.

6. The area of a certain polygon is 5 square feet. Find the area of the similar polygon whose perimeter is in the ratio of m to n to that of the given polygon.

SHEFFIELD SCIENTIFIC SCHOOL, *September*, 1890.

1. Define *locus*. Find the locus of all points equidistant from two parallel lines, proving result.

2. The common chord of two circles is perpendicular to, and is bisected by, the line joining their centres.

3. To divide a given line in extreme and mean ratio. What regular inscribed polygons may be constructed by means of this division? Prove your statement.

4. Two circles are tangent internally, the ratio of their radii being 2 : 3. Compare their areas, and also the area left in the larger circle with each.

5. Of all polygons formed of given sides the maximum may be inscribed in a circle.

PRINCETON COLLEGE, *June*, 1890.

If you offer two books of Euclid answer the first five questions only. If you offer all of Plane Geometry OMIT *questions* 3 *and* 4.

1. Define surface, plane, angle, isosceles triangle, proposition, theorem, problem, corollary.

2. Under what conditions are two triangles equal? Prove that two triangles are equal if the three sides of one are equal respectively to the three sides of the other.

For those offering two books of Euclid.

3. The opposite sides and angles of a parallelogram are equal to one another and the diagonal bisects the parallelogram.

4. If a straight line be divided into two equal parts and also into two unequal parts, the rectangle contained by the unequal parts, together with the square on the line between the points of section, is equal to the square on half the line.

5. To construct a square that shall be equal to a given rectilinear figure.

For those offering Plane Geometry.

6. To find a mean proportional between, and a third proportional to, two given straight lines.

7. Inscribe a regular decagon in a given circle.

PRINCETON SCHOOL OF SCIENCE, *June*, 1890.

1. Prove that two angles whose sides are perpendicular each to each are either equal or supplementary.

Define supplementary angles.

2. Prove that if two sides of a triangle are unequal, the angles opposite are unequal, and the greater angle is opposite the greater side.

Define the various kinds of triangles.

3. Prove that in the same or equal circles equal chords are equally distant from the centre.

Define chord, sector, segment, in a circle.

4. Explain how to bisect a given angle. Also how to construct a triangle with three given sides.

5. Prove that if any chord is drawn through a fixed point within a circle, the product of its segments is constant, in whatever direction the chord is drawn.

6. Show how to find a mean proportional between two given lines; and prove the proposition on which the construction depends.

7. Explain how to construct a triangle equivalent to a given polygon.

8. Prove that the perimeters of two regular polygons of the same number of sides are to each other as the radii of their inscribed circles.

Cornell University, *June*, 1890.

1. Define: a broken line, a corollary, a right angle, a geometric locus, a segment of a circle, two similar polygons.

2. If two angles have their sides parallel each to each, they are equal or supplementary.

3. If a quadrilateral be circumscribed about a circle, the sum of two opposite sides is equal to the sum of the other two sides.

4. If through any point, within a circle or without it, two straight lines be drawn to cut the circle, the product of the two segments of the one chord is equal to the product of the two segments of the other chord.

5. Draw a hexagon having one re-entrant angle, and construct a triangle equivalent to this polygon.

6. Of all isoperimetric figures the circle has the greatest area.

Cornell University, *September*, 1890.

1. Define: a straight line, a theorem, two adjacent plane angles, two parallel straight lines, a sector of a circle, two similar triangles.

2. Two parallel straight lines are everywhere equally distant.

3. An angle inscribed in a circle is measured by half the intercepted arc.

4. If the sides of a triangle be cut by a straight line parallel to the base, the segments of the two sides are proportional. Two cases.

5. Draw a pentagon and construct a square equivalent to it.

6. Of all triangles formed with the same two sides, that in which these two sides are perpendicular to each other has the greatest area.

COLUMBIA COLLEGE, *June*, 1890.

1. Show what the sum of the exterior angles of a polygon is equal to.

2. Show that if two quantities be increased or diminished by like parts of each, the results will be proportional to the quantities themselves.

3. Show that through any three points, not in the same straight line, one circumference may be made to pass, and but one.

4. (*a*) Show what the sum of the opposite angles of a quadrilateral inscribed in a circle is equal to.

(*b*) Given, two sides of a triangle and the angle opposite one of them ; construct the triangle.

5. Show that, in any triangle, a line drawn parallel to the base divides the other sides proportionally.

6. Show that if two chords intersect in a circle, their segments will be reciprocally proportional.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,

June, 1890.

1. Two angles whose sides are perpendicular each to each are either equal or supplementary.

2. If two triangles have two sides of one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, the included angle of the first is greater than the included angle of the second.

3. If the diagonals of a quadrilateral are equal, and bisect each other at right angles, the figure is a square.

4. In equal circles, or in the same circle, the greater of two chords subtends the greater arc.

5. The length of the straight line joining the middle points of the non-parallel sides of a circumscribed trapezoid is equal to one-fourth the perimeter of the trapezoid.

6. Two triangles are similar when they are mutually equiangular.

7. The areas of two similar triangles are to each other as the squares of their homologous sides.

8. If the radius of a circle is 6, what is the area of a segment whose arc is 60°? (Take $\pi = 3.1416$.)

HARVARD COLLEGE, *June*, 1891.

In solving problems use for π *the approximate value* $3\frac{1}{7}$.

1. Prove that in an isosceles triangle the angles opposite the equal sides are equal to each other.

The area of a certain isosceles triangle is 50 square feet and each of its equal sides is 10 feet long; find the angles of the triangle.

2. Prove that in any quadrilateral circumscribed about a circle the sum of two opposite sides is equal to the sum of the other two opposite sides.

One vertex of a circumscribed quadrilateral and the directions of the sides which meet at that vertex are given; what is the locus of the centre of the inscribed circle?

3. From a fixed point O of a given circumference are drawn two chords, OP, OQ, so as to make equal angles with a fixed chord, OR, between them. Prove that PQ will have the same direction whatever the magnitude of the equal angles.

4. The diagonals of a certain trapezoid, which are 8 and 12 feet long respectively, divide each other into segments which in the case of the shorter diagonal are 3 feet and 5 feet long. What are the segments of the other diagonal?

5. Assuming that as the number of sides of a circumscribed polygon is indefinitely increased, the perimeter approaches as a limit the circumference of the circle,

and the area of the polygon the area of the circle; prove that the area of a circle is numerically equal to one-half the product of its radius by its circumference.

6. A regular hexagon, the perimeter of which is 42 inches, is inscribed in a circle; what is the area of this circle?

HARVARD COLLEGE, *September*, 1891.

In solving problems use for π the approximate value $3\frac{1}{7}$.

1. Prove that, if two angles of a triangle are equal, the sides opposite these angles are equal to each other.

In a certain isosceles triangle ABC, $A = B = 30°$, and $AC = 20$ inches; show that the perpendicular dropped from C upon AB is 10 inches long.

2. Prove that, in any quadrilateral inscribed in a circumference, opposite angles are supplementary.

What is the locus of the centre of a circumference which passes through two fixed vertices of a given quadrilateral?

3. In a certain triangle ABC, $\overline{AC}^2 - \overline{BC}^2 = \frac{1}{2}\overline{AB}^2$; show that a perpendicular dropped from C upon AB will divide the latter into segments which are to each other as 3 to 1.

4. Prove that, if in the triangle ABC, the line drawn from the vertex C to the middle of the opposite side is equal to half the latter, the area of the triangle is numerically equal to half the product of AC by BC.

5. A regular polygon of n sides is circumscribed about a certain circumference in which another such polygon is inscribed. What is the limit of the ratio of these polygons as n is made larger and larger?

The area of a certain circle is 154 square inches; what angle at the centre is subtended by an arc of the circumference $5\frac{1}{2}$ inches long?

YALE COLLEGE, *June*, 1891.

1. Prove that two straight lines which are both parallel to a third straight line are parallel to each other.

2. Prove that, if from any point without a circle there be drawn a line tangent to the circle and also a line cutting it and not passing through the centre, the tangent is a mean proportional between the whole secant and its external segment.

3. Construct an equilateral triangle whose perimeter shall be 12 inches, and inscribe a circle in it.

NOTE. Write nothing in connection with this problem.

4. Prove that if the points of tangency of the above inscribed circle be joined, the triangle formed is equilateral, and its perimeter is 6 inches.

5. (*a*) Define a geometrical locus, and give examples of loci from the figure drawn in number 3.

(*b*) Define the limit of a variable, and write the caption of some proposition whose proof depends on the theory of limits.

SHEFFIELD SCIENTIFIC SCHOOL, *June*, 1891.

1. In what various ways may systems of three lines be drawn *similarly* in a triangle to meet in one point? Prove any one of your statements.

2. How is the angle between the secants to a circle measured? between a tangent and a chord? Give demonstrations of answers.

3. If through any point within a circle a chord be drawn, the product of the segments formed by the point is constant in whatever direction the chord be drawn.

4. To construct a square whose area is *double* that of a given square.

5. Regular polygons of the same number of sides are similar.

SHEFFIELD SCIENTIFIC SCHOOL, *September*, 1891.

1. Mutually equilateral triangles are equal in all respects.

2. The common chord of two circles is bisected perpendicularly by the line of centres. What does the proposition become when the circles touch?

3. State and prove the theorems possible when a perpendicular is let fall upon the hypotenuse of a right-angled triangle from the opposite vertex.

4. Compute the area of a regular hexagon whose side is 5 feet. Construct a triangle of equivalent area.

5. The square on the side opposite any acute angle of a triangle is equivalent to the sum of the squares on the other two sides diminished by twice the rectangle on one of those sides and the projection of the other upon it.

PRINCETON COLLEGE, *June*, 1891.

1. Prove that if two triangles have two sides of the one equal to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first will be greater than the third side of the second.

Also prove the converse.

2. Prove that a straight line perpendicular to a diameter of a circle at its extremity is tangent to the circle.

Define tangent, secant, diameter, radius, chord, sector.

3. Prove that two mutually equiangular triangles are similar.

What are similar polygons?

4. Show how to construct a square equivalent to the difference of two given squares.

5. Prove that a circle may be circumscribed about, and a circle may be inscribed in any regular polygon.

What is a regular polygon?

6. Prove how to inscribe a regular hexagon in a given circle.

CORNELL UNIVERSITY, *June*, 1891.

1. Define and illustrate by figures, a quadrilateral, a regular hexagon, vertical angles, and the altitude of a triangle.

2. Prove that any side of a triangle is less than the half-sum of all the sides.

3. The line joining the middle points of two sides of a triangle is parallel to the third side and equal to one-half of it.

4. The two tangents to a circle from an outside point are equal.

5. If two triangles have an angle of one equal to an angle of the other they are to each other as the product of the sides including the equal angles.

6. From the obtuse angle of a triangle draw a line which divides the opposite side proportionately to the adjacent sides.

7. The area of a rhombus is equal to one-half the product of its diagonals.

CORNELL UNIVERSITY, *September*, 1891.

1. Define and illustrate by figures supplementary and complementary angles, similar polygons. Give all the conditions of equality of triangles.

2. The perpendiculars from the vertices of a triangle to the opposite sides meet in a point.

3. The bisectors of the angles of a rectangle form a square.

4. The opposite angles of an inscribed quadrilateral are supplementary.

5. If two chords intersect the product of the segments of one is equal to the product of the segments of the other.

6. Prove that if the radius be unity, the apothem of a regular inscribed decagon is $\frac{1}{4}\sqrt{10+2\sqrt{5}}$.

7. Inscribe a square in a scalene triangle.

COLUMBIA COLLEGE, *June*, 1891.

1. If from any point within a triangle two straight lines be drawn to the extremities of any side, the sum will be less than that of the two remaining sides of the triangle. Demonstrate.

2. Prove that if four quantities are in proportion, they will be in proportion by division.

3. (*a*) Given two sides of a triangle, and an angle opposite one of them, to construct a triangle.

(*b*) Find the centre of a given circumference.

4. Prove that any two rectangles are to each oth the products of their bases and altitudes.

5. Prove that, in any triangle, a line drawn parallel to the base divides the other sides proportionally.

6. Prove that triangles which have an angle in each equal are to each other as the rectangles of the including sides.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,

June, 1891.

1. Two triangles are equal when a side and two adjacent angles of one are equal respectively to a side and two adjacent angles of the other.

2. The perpendiculars from the vertices of a triangle to the opposite sides meet in a common point.

3. The interior angle of a regular polygon exceeds the exterior angle by 120°. How many sides has the polygon?

4. The angle between two tangents to a circle is measured by one-half the difference of the intercepted arcs.

5. A, B, and C are any three points on the circumference of a circle, and D and E are the middle points of the arcs AB and AC respectively. If the chord DE cuts the chord AB at F, and the chord AC at G, prove that the angles AFG and AGF are equal.

6. In any triangle the product of any two sides is equal to the diameter of the circumscribed circle multiplied by the perpendicular drawn to the third side from the vertex of the opposite angle.

7. Two triangles having an angle of one equal to an angle of the other, are to each other as the products of the sides including the equal angles.

8. If the circumference of a circle is divided into any number of equal arcs, their chords form a regular polygon inscribed in the circle.

Massachusetts Institute of Technology,
September, 1891.

1. The perpendicular is the shortest line which can be drawn from a point to a straight line.

2. The straight line joining the middle points of two sides of a triangle is parallel to the third side and equal to one-half of it.

3. Two parallel tangents intercept equal arcs on the circumference of a circle.

4. If AD and BE are the perpendiculars from the vertices A and B, respectively, of the triangle ABC, to the opposite sides, and intersect at O, prove that $OA : OE = OB : OD$.

5. Two triangles are similar when the sides of one are perpendicular, respectively, to the sides of the other.

6. In an obtuse-angled triangle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides, increased by twice the product of one of these sides and the projection of the other side upon it.

7. The area of a trapezoid is equal to one-half the sum of its parallel sides multiplied by its altitude.

8. If the diagonals AC and BG of the regular octagon $ABCDEFGH$ intersect at O, how many degrees are there in the angle AOB?

HARVARD COLLEGE, *June*, 1892.

In solving problems use for π the approximate value $3\frac{1}{7}$.

1. Prove that if two sides of a triangle are unequal, the angle opposite the greater side is greater than the angle opposite the lesser side.

In a certain right triangle one of the legs is half as long as the hypotenuse; what are the angles of the triangle?

2. Show how to find on a given indefinitely extended straight line in a plane, a point O which shall be equidistant from two given points A, B in the plane. If A and B lie on a straight line which cuts the given line at an angle of 45° at a point 7 inches distant from A and 17 inches from B, show that OA will be 13 inches.

3. Prove that an angle formed by a tangent and a chord drawn through its point of contact is the supplement of any angle inscribed in the segment cut off by the chord. What is the locus of the centre of a circumference of given radius which cuts at right angles a given circumference?

4. Show that the areas of similar triangles are to each other as the squares of the homologous sides.

5. Prove that the square described upon the altitude of an equilateral triangle has an area three times as great as that of a square described upon half of one side of the triangle.

6. Find the area included between a circumference of radius 7 and the square inscribed within it.

HARVARD COLLEGE, *September*, 1892.

In solving problems use for π the approximate value $3\frac{1}{7}$.

1. Prove that if two angles of a triangle are unequal, the side opposite the greater angle is greater than the side opposite the lesser angle. The angles of a certain right triangle are to each other in the ratio of the numbers 1, 2, and 3; show that one of the sides of the triangle is half as long as the hypotenuse.

2. Show that the angle included between the internal bisector of one base angle of a triangle and the external bisector of the other base angle is equal to half the vertical angle of the triangle.

3. Prove that the tangents drawn to two intersecting circles from any point on their common chord produced are of equal length.

4. Show that the area of a trapezoid is equal to the product of its altitude by half the sum of its parallel bases, and that the area of any trapezoid is to the area of any similar trapezoid as the product of the two non-parallel sides in the first is to the product of the corresponding sides in the second.

5. Prove that the area of a circle is equal to half the product of its circumference by its radius.

6. The area of a certain regular hexagon is $294\sqrt{3}$ square inches; find the area and the circumference of the circumscribed circle.

YALE COLLEGE, *June*, 1892.

1. Construct accurately by ruler and compass a parallelogram $ABCD$, having the angle A 45°, the side AB 6 units in length, and the altitude 3 of the same units.

Calculate the length of AC.

2. (*a*) State the converse of the following proposition: If a triangle is isosceles, and if a straight line is drawn through the vertex parallel to the base, it bisects an exterior angle of the triangle.

(*b*) Prove the converse as you have stated it.

Make the demonstration as full and clear as possible.

3. Prove two of the following propositions; the work may be limited to drawing a figure and giving a synopsis of the demonstration.

(*a*) If the area of a regular polygon is equal to the product of the perimeter by one-half the apothem, it follows that the area of a circle is equal to πR^2.

(*b*) If two lines are drawn through the same point across a circle, the products of the two distances on each line from this point to the circumference are equal to each other.

(*c*) If the radius of a circle be divided in extreme and mean ratio, the greater segment is equal to one side of a regular inscribed decagon.

SHEFFIELD SCIENTIFIC SCHOOL, *June*, 1892.

1. Prove the two propositions relating to the sum of the *interior* angles of a convex polygon, and the sum of the *exterior* angles formed by producing each side in one direction.

2. In a circle the greater chord subtends the greater arc, and conversely.

3. When is a line said to be divided *harmonically?* From the point P without a circle a secant through the centre is drawn cutting the circle in A and B. Tangents are drawn from P and the points of contact connected by a line cutting AB in Q. Show that P and Q divide AB harmonically.

4. Derive an expression for the area of a regular polygon.

5. When two sides of a triangle are given, at what angle must they intersect if the area shall be a maximum? Prove your answer.

SHEFFIELD SCIENTIFIC SCHOOL, *September*, 1892.

1. Enumerate and define the various sorts of symmetry. Show that two lines symmetrical with respect to a centre are equal and parallel.

2. When two parallel lines are cut by any transversal, the alternate-interior angles are equal.

3. The product of two sides of a triangle is equal to the perpendicular upon the third side into the diameter of the circumscribed circle. How may this be stated as a theorem in areas?

4. Given two similar polygons, construct a third similar to either and equivalent to their sum.

5. Define *similar polygons.* Triangles whose sides are proportional are similar.

PRINCETON COLLEGE, *June,* 1892.

1. Prove the relation between two angles whose sides are perpendicular each to each. When is one line perpendicular to another?

2. Prove what the exterior angles of a convex polygon, made by producing each of its sides in succession, are together equal to.

What is a *convex* polygon?

3. Prove how to inscribe a circle in a given triangle and how to circumscribe a circle about a given triangle.

4. Prove what the square of the side opposite an obtuse angle in any triangle is equal to.

5. Prove how to construct a square equivalent to a given parallelogram.

Define a square, a parallelogram, equivalent figures.

6. Prove what the area of a circle is equal to, and express it in terms of π and R. Give the numerical value of π.

7. Prove that of all isoperimetric polygons of the same number of sides, the maximum polygon is equilateral.

CORNELL UNIVERSITY, *June,* 1892.

1. Explain, with examples, *reductio ad absurdum,* converse, and corollary.

2. The straight lines drawn from the vertices of any triangle, perpendicular to the opposite sides, meet in a point.

3. Angles formed by secants meeting without the circle are measured by half of the difference of the intercepted arcs.

4. *ABC* is an isosceles triangle, having *AB* equal to *AC*. The angle *ABC* is bisected by the straight line *BD*, and the angle *ACB* by the straight line *CD*. Prove that *DB* is equal to *DC*.

5. The diagonals of a rhombus are unequal, but bisect each other at right angles.

6. Describe a circle which shall touch a given circle, and touch a given straight line at a given point. How many solutions are there; and what is the nature of the contact when the given line (1) falls without, (2) cuts the given circle?

7. A ball of lead, 4 inches in diameter, is covered with gold. Find the thickness of the gold, in order that (1) the volumes of gold and lead may be equal, (2) the surface of the gold may be twice that of the lead.

CORNELL UNIVERSITY, *September*, 1892.

1. Two parallel lines are everywhere equally distant.

2. If two sides of a triangle are unequal, the angles opposite to them are unequal, and the greater angle is opposite to the greater side.

3. Two triangles are similar, when their homologous sides are proportional.

4. Upon a given straight line, to describe a segment which shall contain a given angle.

5. On a given straight line, to construct a polygon similar to a given polygon.

6. To inscribe a regular decagon in a given circle.

7. Every point in the bisector of an angle is equally distant from the sides of the angle; and every point not in the bisector, but within the angle, is unequally distant from the sides of the angle.

COLUMBIA COLLEGE, *June*, 1892.

1. Prove that, in any triangle, the greater side is opposite the greater angle; and conversely, the greater angle is opposite the greater side.

2. Show that in any continued proportion, the sum of the antecedents is to the sum of the consequents as any antecedent to its corresponding consequent.

3. Prove that, in equal circles, incommensurable angles at the centre are proportional to their intercepted arcs.

4. Prove what the square of a side opposite the obtuse angle of a plane triangle is equal to.

5. Prove that, in any plane triangle, the straight line which bisects the angle at the vertex divides the base into segments proportional to the adjacent sides.

6. Find the side and the altitude of an equilateral triangle in terms of the radius of the inscribed circle.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,

June, 1892.

1. Two triangles are equal when the three sides of one are equal respectively to the three sides of the other.

2. In the same circle, or in equal circles, equal chords are equally distant from the centre.

3. If AC and BC are tangents to a circle whose centre is O, from a point C without the circle, prove that the centre of the circle which passes through O, A, and B, bisects OC.

4. Two triangles are similar when their homologous sides are proportional.

5. Two triangles having an angle of one equal to an angle of the other are to each other as the products of the sides including the equal angles.

6. Define a segment of a circle; equivalent triangles. When are two polygons called mutually equilateral?

Two sides of a triangle are 17 and 10; the perpendicular from their intersection to the third side is 8; what is the length of the third side?

7. AD and BC are the parallel sides of the trapezoid $ABCD$, whose diagonals intersect at O. Prove

$$\text{area } AOD : \text{area } BOC = \overline{AO}^2 : \overline{OC}^2.$$

8. If the circumference of a circle is divided into any number of equal arcs, their chords form a regular po' gon inscribed in the circle.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,

September, 1892.

1. If two straight lines are drawn from a point within a triangle to the extremities of any side, the angle included by them is greater than the angle included by the other two sides.

2. The perpendiculars from the vertices of a triangle to the opposite sides meet in a common point.

3. In the same circle, or in equal circles, the less of two chords is at the greater distance from the centre.

4. An angle formed by a tangent and a chord is measured by one-half its intercepted arc.

5. The diagonals AC and BD of the quadrilateral $ABCD$ intersect at E. If $AE : EC = BE : ED$, prove that the triangles AEB and DEC are similar.

6. If through a fixed point without a circle a secant and a tangent are drawn, the product of the whole secant and its external segment is equal to the square of the tangent.

7. In an obtuse-angled triangle the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides, increased by twice the product of one of these sides and the projection of the other side upon it.

8. A regular hexagon $ABCDEF$ is inscribed in a circle whose radius is 2; find the length of the diagonal AC.

Define an inscribed angle; a diagonal of a polygon; similar triangles.

HARVARD COLLEGE, *June*, 1893.

In solving problems use for π the approximate value $3\frac{1}{7}$.

1. Prove that two oblique lines drawn from a given point to a given line are equal if they meet the latter at equal distances from the foot of the perpendicular dropped from the point upon it.

How many lines can be drawn through a given point in a plane so as to form in each case an isosceles triangle with two given lines in the plane?

2. Prove that in the same circle, or in equal circles, equal chords are equally distant from the centre, and that of two unequal chords the less is at the greater distance from the centre.

Two chords of a certain circle bisect each other. One of them is 10 inches long; how far is it from the centre of the circle? What is the radius of the circle?

A variable chord passes, when produced, through a fixed point without a given circle. What is the locus of the middle point of the chord?

3. A common tangent of two circumferences which touch each other externally at A, touches the two circumferences at B and C respectively; show that BA is perpendicular to AC.

4. Assuming that the areas of two triangles which have an angle of the one equal to an angle of the other are to each other as the products of the sides including the equal angles, prove that the bisector of an angle

a triangle divides the opposite side into parts which are proportional to the sides adjacent to them.

5. Prove that the circumference of two circles have the same ratio as their radii.

6. A quarter-mile running track consists of two parallel straight portions joined together at the ends by semicircumferences. The extreme length of the plot enclosed by the track is 180 yards. Find the cost of sodding this plot at a quarter of a dollar per square yard.

HARVARD COLLEGE, *September*, 1893.

In solving problems use for π the approximate value $3\frac{1}{7}$.

1. Prove that two equal oblique lines drawn from a given point to a line meet the line at equal distances from the foot of the perpendicular dropped from the given point upon it.

How many straight lines can be drawn in a plane so as to be equally distant from three given points in the plane not on one straight line?

2. Prove that in the same circle, or in equal circles, chords equally distant from the centre are equal, and that of two chords unequally distant from the centre that is the greater which is nearer the centre.

In a certain circle a chord is 10 inches long while another chord twice as far from the centre as the first is 5 inches long; find the radius of the circle and the distances of the chords from the centre.

A variable secant passes through a fixed point without a given circle ; what is the locus of the middle point of the whole secant ?

3. Two circumferences intersect at A and B. Through B any secant is drawn so as to cut the circumferences in C and D respectively. Show that the angle CAD is the same for all secants drawn through B. What value has this angle when the circumferences intersect each other orthogonally ?

4. Assuming that the area of two triangles which have equal altitudes are to each other as the bases when the latter are commensurable, prove that the same proportionality exists when they have no common measure.

5. Show how to bisect the area of a triangle by a line parallel to its base, but do not take time to prove that your construction is correct.

6. A stone bridge 20 feet wide has a circular arch of 140 feet span at the water level. The crown of the arch is $140\,(1 - \frac{1}{2}\sqrt{3})$ feet above the surface of the water. How many square feet of surface must be gone over in cleaning so much of the under side of the arch as is above water ?

YALE COLLEGE, *June*, 1893.

1. Prove that if the diagonals of a quadrilateral bisect each other the figure is a parallelogram.

2. Prove that in any right-angled triangle the square on the side opposite to the right angle is equal to the sum of the squares on the other two sides.

A purely geometrical proof is preferred.

State fully each principle employed in the proof.

3. Given a straight line AB of indefinite length, and a point C without it. Find a point in AB equally distant from A and C.

Make the necessary construction accurately with ruler and compass.

In what case is the solution impossible?

4. Given an angle COD at the centre of a circle and the line CA meeting the circumference in B, and DO produced in A, so that AB is equal to the radius of the circle. Prove that the angle A is equal to one-third of the angle COD.

SHEFFIELD SCIENTIFIC SCHOOL, *June*, 1893.

1. Enumerate and define the different sorts of quadrilaterals. Show that if the middle points of the non-parallel sides of a trapezoid be joined by a straight line, this line is parallel to the other sides and equal to one-half their sum.

2. Give constructions for the ***inscribed***, ***escribed***, and ***circumscribed*** circles of any triangle.

3. (*a*) Two polygons are similar when composed of the same number of triangles, similar each to each, and similarly placed.

(*b*) When the areas of two similar polygons are in the ratio of m to n, in what ratio are the homologous sides?

4. In the triangle whose sides are a, b, and c, determine the segments of each side made by the bisector of the opposite angle.

5. A circle may be inscribed in and circumscribed about a regular polygon.

SHEFFIELD SCIENTIFIC SCHOOL, *September*, 1893.

1. What are equal angles, complementary angles, supplementary angles? Show that angles whose sides are parallel or perpendicular are equal or supplementary.

2. If through a point without a circle a secant be drawn, the product of the whole secant and the part without the circle is constant, in whatever direction the secant is drawn. State the corresponding theorem when the point is inside the circle.

3. What is it to *measure* a quantity? When are two magnitudes *commensurable;* when *incommensurable?* When a line is drawn parallel to any side of a triangle, show that it divides the other two sides proportionally.

4. Two triangles have sides a, b, c and a', b', c'; the angles opposite a and a' are equal; find the ratio of their areas.

5. Show that in a series of regular polygons of equal *areas*, the perimeters decrease as the number of sides increases.

PRINCETON COLLEGE, *June*, 1893.

1. Define parallel lines. State the parallel postulate (or axiom).

Prove that if a straight line meet two parallel straight lines, then the alternate angles are equal. State and prove the converse.

2. Two triangles are equal if the three sides of the one are equal respectively to the three sides of the other.

3. Equal chords in a circle are equidistant from the centre; and conversely.

4. The bisector of an exterior angle of a triangle meets the opposite side produced at a point the distances of which from the two extremities of this side are proportional to the other two sides.

5. To construct a square equal to a given parallelogram.

State the difference in the meaning of the word equal here and in question 2.

6. The perimeters of two regular polygons of the same number of sides are to each other as their radii or as their apothems.

Define regular polygons, their radii, their apothems.

CORNELL UNIVERSITY, *June*, 1893.

1. Inscribe a square in a scalene triangle.

2. From one vertex of a triangle draw a line to the opposite side which shall be a mean proportional between the segments of that side.

3. How many sides has the polygon the sum of whose exterior angles is double that of its interior angles ?

4. Upon a given straight line describe a segment which shall contain a given angle.

5. Through a point in the circumference of a circle chords are drawn. Find the locus of their middle points.

6. Given $\frac{\text{[illegible]}}{x} = \frac{x}{3}$. Construct x.

7. Show that an equiangular polygon inscribed in a circle is regular if the number of its sides is odd.

CORNELL UNIVERSITY, *September*, 1893.

1. What is the greatest number of re-entrant angles a polygon may have compared to the number of its sides ? What is the value of the re-entrant angles of a pentagon in terms of the interior angles not adjacent ?

2. The sum of the angles of a triangle is equal to two right angles.

3. From a point outside of a circle secants are drawn. Find the locus of their middle points.

4. If two chords intersect in a circle the product of the segments of one is equal to the product of the segments of the other.

5. Two triangles having an angle of one equal to an angle of the other are to each other as the product of the sides including the equal angles.

6. Construct a polygon similar to a given polygon and having two and a half times its area.

7. Through a point in one side of a triangle draw a line parallel to the base which shall bisect the area of the triangle.

COLUMBIA COLLEGE, *June*, 1893.

1. If from any point within a triangle two straight lines are drawn to the extremities of any side, their sum is less than that of the two remaining sides of the triangle. Dem.

2. An inscribed angle is measured by what? Dem.

3. Given two sides of a triangle and the angle opposite one of them, construct the triangle and explain construction.

4. What is the square described on the hypotenuse of a right-angled triangle equal to? Dem.

5. Show that if two chords intersect in a circle, their segments are reciprocally proportional.

6. Construct a mean proportional between two straight lines, and explain construction.

7. If the radius of a circle is divided in extreme and mean ratio, the greater segment is equal to one side of a regular inscribed decagon. Dem.

8. The side of an equilateral triangle inscribed in a circle is $8\sqrt{3}$; find area of circle.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY, *June*, 1893.

1. If oblique lines are drawn from a point to a straight line, two oblique lines cutting off equal distances from the foot of the perpendicular are equal; and of two oblique lines cutting off unequal distances from the foot of the perpendicular the more remote is the greater.

2. The straight line joining the middle points of the non-parallel sides of a trapezoid is parallel to the bases and equal to one-half their sum.

3. In equal circles, or in the same circle, the greater of two chords subtends the greater arc.

4. State and prove the converse of the preceding.

If the diameter of a circle is 3 inches, what is the length of an arc of 80° ?

5. If AB and CD are equal chords of a circle and intersect at E, prove that $AE = ED$ and $BE = EC$.

6. If any two chords are drawn through a fixed point in a circle, the product of the segments of one is equal to the product of the segments of the other.

7. AD and BC are the parallel sides of a trapezoid $ABCD$, whose diagonals intersect at E. If F is the

middle point of *BC*, prove that *EF* produced bisects *AD*.

8. Two similar triangles are to each other as the squares of their homologous sides.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,

September, 1893.

1. If two sides of a triangle are unequal, the angles opposite are unequal, and the greater angle lies opposite the greater side.

2. State and prove the converse of the preceding.

Define a convex polygon; a segment of a circle; equivalent polygons; similar sectors.

3. An inscribed angle is measured by one-half its intercepted arc.

4. If *E* and *F* are the middle points of the non-parallel sides *AB* and *CD*, respectively, of the trapezoid *ABCD*, and *EF* cuts *AC* at *G* and *BD* at *H*, prove $EG = FH$.

5. A straight line which divides two sides of a triangle proportionally is parallel to the third side.

6. *AB* and *AC* are tangents at the points *B* and *C* to the circumference whose centre is *O*; prove that the angle *OBC* is one-half the angle *A*.

7. The area of a parallelogram is equal to the product of its base and altitude.

8. The perimeters of two regular polygons are to each other as the radii of their circumscribed, or of their inscribed circles; and their areas are to each other as the squares of these radii.

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